SOME DYNAMIC AIRFRAME STABILITY CONSIDERATIONS IN THE CONTROL PROBLEM

ROBERT BRUCE BORTHWICK

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Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
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ABSTRACT

The dynamic stability analysis of the airframe for consideration in the control problem was made considering six degrees of freedom. The intention was to explore a method and determine the extent that such a scheme could be used in a system known to be non-linear at larger disturbances. It was determined that a very workable system exists whereby differential equations may be simply converted into block diagrams and vice versa. The system is workable for various flight conditions. It can also handle disturbances up to one tenth radian in most cases but is restricted to linear approximations.

The system is compatible to any multi-loop control problem and is not restricted to airframe control. A standardized form now exists upon which multi-loop compensation theory research may be conducted using either basic hardware or differential equations as the basic system.

The graduate work, for which this thesis is a partial requirement, was performed while the author was attending the U. S. Naval Postgraduate School during assignment to the Aeronautical Engineering curriculum (Avionics).

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USE OF SYMBOLS

U	$\triangleq$	x component of velocity (scalar)
V	$\triangleq$	y component of velocity (scalar)
W	$\triangleq$	= z component of velocity (scalar)
A	$\triangleq$	moment of inertia about x axis, slug-ft^2 (whole mass)
B	$\triangleq$	moment of inertia about y axis, slug-ft^2 (whole mass)
C	$\triangleq$	moment of inertia about z axis, slug-ft^2 (whole mass)
D	$\triangleq$	product of inertia, $\int yz\,dm$, slug-ft^2
E	$\triangleq$	product of inertia, $\int xz\,dm$, slug-ft^2
F	$\triangleq$	product of inertia, $\int xy\,dm$, slug-ft^2
P	$\triangleq$	angular component of rotation about x-axis (scalar) radian/sec
Q	$\triangleq$	angular component of rotation about y-axis (scalar) radian/sec
R	$\triangleq$	angular component of rotation about z-axis (scalar) radian/sec
K_1	$\triangleq$	relative angular velocity of rotor term on x-axis (scalar)
K_2	$\triangleq$	relative angular velocity of rotor term on y-axis (scalar)
K_3	$\triangleq$	relative angular velocity of rotor term on z-axis (scalar)
a	$\triangleq$	moment of inertia of rotor along x-axis
b	$\triangleq$	moment of inertia of rotor along y-axis
c	$\triangleq$	moment of inertia of rotor along z-axis
a_c	$\triangleq$	acceleration of airframe at its mass center ft/sec^2
m	$\triangleq$	mass of control surface
e	$\triangleq$	eccentricity of control surface between hinge and the control mass center
ϕ	$\triangleq$	angular displacement about the reference (x_*) axis
$\dot{\phi}$	$\triangleq$	angular velocity about the reference (x_*) axis

$\theta \triangleq$ angular displacement about the reference (y_*) axis

$\dot{\theta} \triangleq$ angular velocity about the reference (y_*) axis

$\psi \triangleq$ angular displacement about the reference (z_*) axis

$\dot{\psi} \triangleq$ angular velocity about the reference (z_*) axis

$$12_p \triangleq \frac{\rho U_o^2 S_r C_r}{2 I_r} \; C_{h_{r_p}}$$

$$12_r \triangleq \frac{\rho U_o^2 S_r C_r}{2 I_r} \; C_{h_{r_r}}$$

$$12_\zeta \triangleq \frac{\rho U_o^2 S_r C_r}{2 I_r} \; C_{h_{r_\zeta}}$$

$$12_{\dot{\zeta}} \triangleq \frac{\rho U_o^2 S_r C_r}{2 I_r} \; C_{h_{r_{\dot{\zeta}}}}$$

$\eta \triangleq$ the elevator position angle (radians)

$\xi \triangleq$ the aileron position angle (radians)

$\zeta \triangleq$ the rudder position angle (radians)

$F_x \triangleq$ force in the direction of the x axis (scalar)

$F_y \triangleq$ force in the direction of the y axis (scalar)

$F_z \triangleq$ force in the direction of the z axis (scalar)

$L \triangleq$ moment about the x axis (scalar)

$M \triangleq$ moment about the y axis (scalar)

$N \triangleq$ moment about the z axis (scalar)

$$X_u \triangleq \frac{\rho SU_o}{2m}\left[(2C_{T_o} + U_oC_{T_u})\cos\Theta_T - (2C_{D_o} + U_oC_{D_u})\right]$$

$$X_w \triangleq \frac{\rho SU_o}{2m}(C_L - C_{D_\alpha})$$

$$X_{\dot w} \cong 0$$

$$X_q \cong 0$$

$$Z_u \triangleq \frac{\rho SU_o}{2m}\left[(2C_{T_o} + U_oC_{T_u})\sin\Theta_T + (2C_{L_o} + U_oC_{L_u})\right]$$

$$Z_w \triangleq -\frac{\rho SU_o}{2m}(C_{L_\alpha} + C_{D_o})$$

$$Z_{\dot w} \triangleq -\frac{\rho SU_oC_{L_{\dot\alpha}}}{4m}$$

$$Z_q \triangleq -\frac{\rho SU_oc}{4m}C_{L_q}$$

$$Z_\eta \triangleq \frac{\rho U^2S}{2m}C_{L_\eta}$$

$$M_u \triangleq \frac{\rho SU_oc}{B}\left[\left(\frac{U}{2}C_{m_u} + C_{m_o}\right) + z_m(2C_{T_o} + U_oC_{T_u})\right]$$

$$M_w \triangleq \frac{\rho SU_oc}{2B}C_{m_\alpha}$$

$$M_{\dot w} \triangleq \frac{\rho Sc^2C_{m_{\dot\alpha}}}{4B}$$

$$M_q \triangleq \frac{\rho USc^2}{4B}C_{m_q}$$

$$M_\eta \triangleq \frac{\rho SU^2c}{2B}C_{m_\eta}$$

$$M_{\dot\eta} \triangleq \frac{\rho S U^2 c_e}{2I} C_{m_{\dot\eta}}$$

$$H_u \triangleq \frac{\rho S U_o^2 c_e}{2\ I_e} C_{h_{eu}}$$

$$H_\alpha \triangleq \frac{\rho S U_o c_e}{2\ I_e} C_{h_{e\alpha}}$$

$$H_{\dot\alpha} \triangleq \frac{\rho S U_o^2 c_e}{2\ I_e} C_{h_{e\dot\alpha}}$$

$$H_q \triangleq \frac{\rho S U_o^2 c_e}{2\ I_e} C_{h_{eq}}$$

$$H_\eta \triangleq \frac{\rho U^2 c S_{ce}}{2\ I_e} C_{h_{e\eta}}$$

$$H_{\dot\eta} \triangleq \frac{\rho U^2 c S_{ce}}{2\ I_e} C_{h_{e\dot\eta}}$$

$$Y_v \triangleq \frac{\rho U_o S C_{y_\beta}}{2m}$$

$$Y_p \triangleq \frac{\rho U_o S b C_{y_p}}{4m}$$

$$Y_r \triangleq \frac{\rho U_o S b C_{y_r}}{4m}$$

$$Y_\zeta \triangleq \frac{1/\rho S U_o^2 C_y}{2m}$$

$$L_v \triangleq \frac{\rho S U_o b C_{l_\beta}}{2I}$$

$$L_p \triangleq \frac{\rho S U_o b^2 C_{l_p}}{4I}$$

$$L_r \triangleq \frac{\rho S U_o b^2 C_{l_r}}{4I}$$

$$L_\xi \triangleq \frac{\rho S U_o^2 b}{2A} \quad C_{l_\xi}$$

$$L_{\dot{\xi}} \triangleq \frac{\rho S U_o^2 b}{2A} \quad C_l \, \xi$$

$$L_\zeta \triangleq \frac{\rho S U_o^2 b}{2A} \quad C_{l_\zeta}$$

$$N_V \triangleq \frac{\rho S U_o b}{2C} \quad C_{n_\beta} \,, \quad N_\beta = U_o N_V$$

$$N_p \triangleq \frac{\rho S U_o b^2}{4C} \quad C_{N_p}$$

$$N_r \triangleq \frac{\rho S U_o b^2}{4C} \quad C_{N_r}$$

$$N_\zeta \triangleq \frac{\rho S U_o b}{2C} \quad C_{n_\zeta}$$

$$H_\rho \triangleq \frac{\rho S U_o^2 b}{2C} \quad C_{\eta_\zeta}$$

$$H_p \triangleq \frac{\rho U_o^2 S_a C_a}{i_a} \quad C_{h_{a_p}}$$

$$H_r \triangleq \frac{\rho U_o^2 S_a C_a}{i_a} \quad C_{h_{a_r}}$$

$$H_\xi \triangleq \frac{\rho U_o^2 S_a C_a}{2 \, i_a} \quad C_{h_a \xi}$$

$$H_{\dot{\xi}} \triangleq \frac{\rho U_o^2 S_a C_a}{2 \, i_a} \quad C_{h_a \dot{\xi}}$$

$$12\rho \triangleq \frac{\rho U_o S_r C_r}{2 \, i_r} \quad C_{h_{r_\beta}}$$

1. Introduction

The types of motion that result from a disturbance in some equilibrium flight condition and the transient motion of the aircraft in response to control movements are analyzed in the study of the dynamic characteristics of an airplane. Over the course of years the effects of dynamic stability in aircraft have been known to designers. The mathematical computations were discussed by early aeronautics pioneers which included Lanchester, Bryant and Glauert. In spite of the fact that dynamic stability effects were known, most people paid little concern to this field. As design groups solved other problems of aircraft design, the dynamic stability effects were automatically solved as a secondary result. However, since World War II, aircraft development has pushed into the field of very high performance aircraft. In many cases the dynamic stability problem doesn't solve itself as happened previously. Furthermore, the fire control problem has entered the scene which has generated a control problem of no small magnitude.

This work will consider the aircraft in six degrees of freedom and explore paths of compensation and solution. It is the prime intention to explore the work of Chu[1] in multi loop servo systems and determine the value of it as applied to the aircraft stability.

The first section will set down a general definition of terms and the basic equations of motion as related to aircraft. It will further set down the basic assumptions upon which the equations are formed.

The equations motion for the airframe which are completely derived in Appendix I are:

$$F_x = (\dot U + QW - RV)m = X - mg \sin \Theta \qquad (I-1)$$

$$F_y = (\dot V + RU - PW)m = Y + mg \cos \Theta \sin \psi \qquad (I-2)$$

$$F_z = (\dot W + PV - QU)m = Z + mg \cos \Theta \cos \psi \qquad (I-3)$$

$$L = A\dot P + a\dot K_1 + (C - B)QR + D(Q^2 - R^2) - E(PQ + \dot R)$$
$$+ FPR + cK_3Q - bK_2R \qquad (I-4)$$

$$M = B\dot Q + b\dot K_2 + (A - C)PR + E(P^2 - R^2) - FQR + DPQ$$
$$+ aK_1R - cK_3P \qquad (I-5)$$

$$N = C\dot R + c\dot K_3 + (B - A)PQ + F(Q^2 - P^2) + E(QR - \dot P)$$
$$- DPR + bK_2P - aK_1Q \qquad (I-6)$$

$$H_e + F_e = I_e \ddot\eta - m_e l_e a_{cz} - P_{ex} (RP - Q) \qquad (I-7)$$

$$H_r + F_r = I_r \ddot\zeta - m_r l_r a_{cy} - (R + PQ)P_{rx} - (RQ - P)rz \qquad (I-8)$$

$$2 H_a + F_a = I_a \ddot\xi + 2 P_{ay} (RQ + P) \qquad (I-9)$$

The angular and linear velocity of a free body in space are

$$P = \dot\phi - \dot\psi \sin \Theta \qquad (I-10)$$

$$Q = \dot\Theta \cos \phi + \dot\psi \sin \phi \cos \Theta \qquad (I-11)$$

$$R = \dot\psi \cos \phi \cos \Theta - \dot\Theta \sin \phi \qquad (I-12)$$

$$\frac{dx^*}{dt} = U \cos \Theta \cos \psi + V (\sin \phi \sin \Theta \cos \psi - \cos \phi \sin \psi)$$
$$+ W (\cos \phi \sin \Theta \cos \psi + \sin \phi \sin \psi) \qquad (I-13)$$

$$\frac{dy^*}{dt} = U \cos \Theta \sin \psi + V (\sin \phi \sin \Theta \sin \psi + \cos \phi \cos \psi)$$
$$+ W (\cos \phi \sin \Theta \sin \psi - \sin \phi \cos \psi) \qquad (I-14)$$

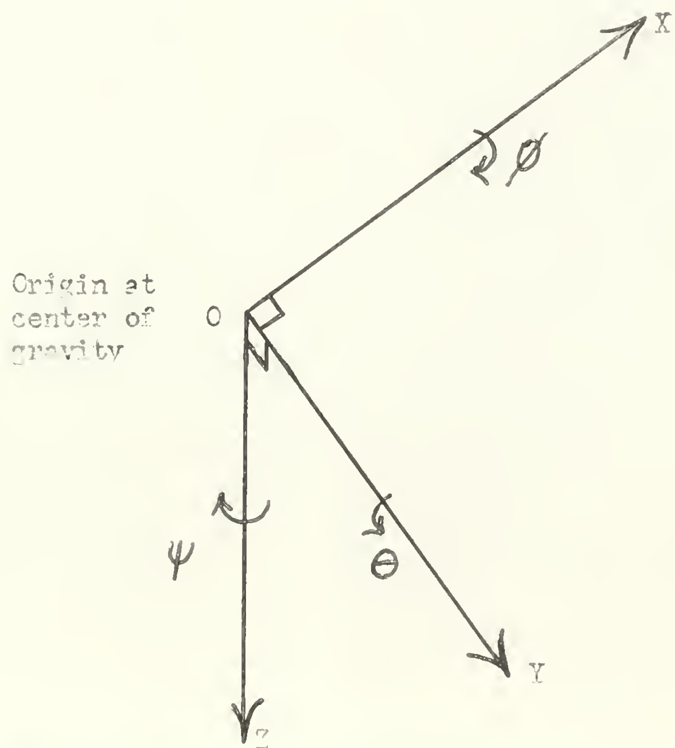
$$\frac{dz^*}{dt} = -U \sin \Theta + V \sin \phi \cos \Theta + W \cos \phi \cos \Theta \qquad (I-15)$$

Consideration of the aerodynamic forces must be resolved into the effects they produce on various parts of the body in some flight condition and also the resultant moment effects they produce.

The first or necessary condition is to establish a reference condition and proceed from there. A reference condition is some flight condition where equilibrium exists in all equations and the summation of forces is zero. This is a steady state condition.

The set of axis upon which this rigid body is oriented is commonly called the Eulerian Axis system. General aerodynamics usage modifies the z axis so the positive direction points downward.

Figure 1-1



NORMAL AIRFRAME AXIS ORIENTATION

The first rotation moves about the Z axis in a direction Ψ and is known as yaw. The next is the movement about the Y axis which is term Θ and known as pitch. The third motion about an axis is roll about the X axis and is maneuvered in terms of ϕ. Caution must be used in determining these positions because the order in using this system is not commutative.

The derivation for the yaw rate R, the roll rate P and the pitch rate Q are shown in detail in Appendix Two. They are:

$$P = \dot{\phi} - \Psi \sin \Theta \qquad \text{(I-16)}$$

$$Q = \dot{\Theta} \cos \phi + \Psi \sin \phi \cos \Theta \qquad \text{(I-17)}$$

$$R = \dot{\Psi} \cos \phi \cos \Theta - \dot{\Theta} \sin \phi \qquad \text{(I-18)}$$

and where small angles are involved

$$P \cong \dot{\phi} \qquad \text{(I-19)}$$

$$Q \cong \dot{\Theta} \qquad \text{(I-20)}$$

$$R \cong \dot{\Psi} \qquad \text{(I-21)}$$

Similarly, Etkin shows:

$$\dot{\Theta} = Q \cos \phi - R \sin \phi \qquad \text{(I-22)}$$

$$\dot{\phi} = P + Q \sin \phi \tan \Theta + R \cos \phi \tan \Theta \qquad \text{(I-23)}$$

$$\dot{\Psi} = (Q \sin \phi + R \cos \phi) \sec \Theta \qquad \text{(I-24)}$$

The general equations of motion have to be applied to the aircraft motion. The effects of axis movement upon the weight force of the aircraft will be shown as well as the reference state condition. It must be remembered that the gravity force is always oriented toward the center of the earth. The reference body axis may or may not be

oriented in this direction.

From the discussion of the Eulerian angle concept, it is determined that a reasonable reference condition would be

$$F_x = O = X_o - mg \sin \Theta_o \tag{I-25}$$

$$Fy = Y_o - W \cos \Theta_o \sin \phi_o \tag{I-26}$$

$$F_z = 0 = Z_o \cos \Theta_o \cos \phi_o \tag{I-27}$$

This allows for a steady state condition where the aircraft is not accelerating but neither is it necessarily in straight and level flight. From the previous illustration of Eulerian angles coupled with the derivations in Appendix II, the gravity forces can be determined as shown below.

$$\begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \cos\Psi\sin\Theta\sin\phi & \cos\Psi\cos\phi & \cos\Theta\sin\phi \\ -\sin\Psi\cos\phi & +\sin\Psi\sin\Theta\sin\phi & \\ \cos\Psi\sin\Theta\cos\phi & \sin\Psi\sin\Theta\cos\phi & \cos\Theta\cos\phi \\ -\sin\Psi\sin\phi & -\cos\Psi\sin\phi & \end{bmatrix} x mg = \begin{bmatrix} my_x \\ my_y \\ \\ my_z \\ \\ \end{bmatrix}$$

Discussion and derivation of this general cosine matrix are found in several sources (2,3). The overall result of the equations of motion is shown below.

$$m(\dot{U} + QW - RV) = X - mg(\sin\Theta_0)(\cos\Theta\cos\Psi) + mg(\cos\Theta_0\sin\phi_0)(\cos\Theta\sin\Psi) - mg(\cos\Theta_0\cos\phi_0)(\sin\Theta\sin\phi) \quad (I\text{-}28)$$

$$m(\dot{V} + RU - PW) = Y - mg(\sin\Theta_0)(\cos\Psi\sin\Theta\sin\phi - \sin\Psi\cos\phi) + mg(\cos\Theta_0\sin\phi_0)(\cos\Psi\cos\phi + \sin\Psi\sin\Theta\sin\phi) + mg(\cos\Theta_0\cos\phi_0)(\cos\Theta\sin\phi) \quad (I\text{-}29)$$

$$m(\dot{W} + PV - QU) = Z - mg(\sin\Theta_0)(\cos\Psi\sin\Theta\cos\phi - \sin\Psi\sin\phi) + mg(\cos\Theta_0\sin\phi_0)(\sin\Psi\sin\Theta\cos\phi - \cos\Psi\sin\phi) + mg(\cos\Theta_0\cos\phi_0)(\cos\Theta\cos\phi) \quad (I\text{-}30)$$

$$L = A\dot{P} + a\dot{K}_1 + QR(C-B) - E(PQ + \dot{R}) + cK_3Q - bK_2R \quad (I\text{-}31)$$

$$M = B\dot{Q} + b\dot{K}_2 + PR(A-C) + E(P^2 - R^2) + aK_1R - cK_3P \quad (I\text{-}32)$$

$$N = C\dot{R} + c\dot{K}_3 + PQ(B-A) + E(QR - \dot{P}) + bK_2P - aK_1Q \quad (I\text{-}33)$$

The above equations are complete except for the external forces X, Y, and Z and the external moments L, M, and N. These forces and moments are composed of aerodynamic and propulsion effects and the moments due to control surfaces. These equations are very general in nature and contain the following assumptions for simplification from the most general and complex case. These assumptions are:

1. The earth is assumed to be fixed in space and the Earth's atmosphere is assumed to be fixed with respect to the earth.
2. The airframe is assumed to be a rigid body.

3. The mass of the body is assumed constant during the time of the problem.

4. For an aircraft or missile, the XZ plane is assumed to have mirror symmetry.

The above equations then cover the six degrees of freedom found in an aircraft or missile. Coupled with the other auxiliary equations covering roll, pitch and yaw effects along with control motions, solution is probably possible. To date, this has been accomplished either by a machine using specific coefficients for various terms or several simplifications. Discussion of this will be conducted in succeeding sections.

2. Consideration of External Forces and Moments on a Body in Motion

In the previous section the equations of motion were discussed. The aerodynamic thrust and control effects were simply grouped in X, Y, Z, L, M and N. It is the aim of this chapter to discuss these in detail with respect to the effect on the dynamic control stability problem.

Consider the term X when the aircraft is in a disturbed condition. The reference condition is X_o and thus $X = X_o + \Delta X$ for some disturbed condition. The question is then to determine the Δ X components.

The force and moment variables can be expressed in a Taylor's Series[6].

$$F = F_o + \left(\frac{\partial F}{\partial \alpha}\right)\alpha + \left(\frac{\partial^2 F}{\partial \alpha^2}\right)\frac{\alpha^2}{2!} --- + \left(\frac{\partial F}{\partial \beta}\right)\beta + \frac{\partial^2 F}{\partial \beta^2}\frac{\beta^2}{2!} \ldots \tag{II-1}$$

For practical reasons the effects of second and higher order terms are omitted from further considerations as will be discussed later. Considering the motion of the aircraft itself the X forces would be

$$X = X_o + \left(\frac{\partial X}{\partial u}\right)u + \left(\frac{\partial X}{\partial \dot{u}}\right)\dot{u} + \left(\frac{\partial X}{\partial w}\right)w + \left(\frac{\partial X}{\partial \dot{w}}\right)\dot{w} + \left(\frac{\partial X}{\partial q}\right)q + \left(\frac{\partial X}{\partial \dot{q}}\right)\dot{q}$$

$$+ \left(\frac{\partial X}{\partial v}\right)v + \left(\frac{\partial X}{\partial \dot{v}}\right)\dot{v} + \left(\frac{\partial X}{\partial p}\right)p + \left(\frac{\partial X}{\partial \dot{p}}\right)\dot{p} + \left(\frac{\partial X}{\partial r}\right)r + \left(\frac{\partial X}{\partial \dot{r}}\right)\dot{r} \tag{II-2}$$

Similarly,

$$Y = Y_o + \left(\frac{\partial Y}{\partial v}\right)v + \left(\frac{\partial Y}{\partial \dot{v}}\right)\dot{v} + \left(\frac{\partial Y}{\partial w}\right)w + \left(\frac{\partial Y}{\partial \dot{W}}\right)\dot{w} + \left(\frac{\partial Y}{\partial r}\right) + \left(\frac{\partial Y}{\partial \dot{r}}\right)\dot{r} \qquad \text{II-3}$$

For convenience all terms will be expressed in letter subscript form. It can readily be seen then that considering X, Y, Z, L, M and N and the effects of U, V, W, P, Q, and R plus the acceleration terms U, V, W, P, Q, and R there are 72 derivative terms required. This does not include the control effects which can add another 36 derivatives. Needless to say, considerable effort has been expended in justifying simplifying assumptions. In addition to this, obtaining reliable coefficients for various derivative terms is not always possible.

It is not the intention of this work to cover the computational methods for these various derivatives. The scope is too large. A search of all authoritative books [3, 4, 5, 6] on the subject yields a table of terms which can be reasonably determined. This appears as Table I. These considerations are mentioned to bring out the awareness of the different causes and effects of external forces. There are considerable gaps in the table. Determination of these to complete the table would be a topic of a complete study in itself. Much work remains to be done in this field.

Furthermore, some of the methods of computing the known stability derivatives are questionable. Some computations from theory

	$u, \dot{u}$	$w, \dot{w}$	$Q, \dot{Q}$	$\eta, \dot{\eta}$	$v, \dot{v}$	$P, \dot{P}$	$R, \dot{R}$	$\xi, \dot{\xi}$	$\zeta, \dot{\zeta}$
ΔX	$\frac{\partial X}{\partial u} u$; ≈ 0	$\left(\frac{\partial X}{\partial w}\right) w$; $\left(\frac{\partial X}{\partial \dot{w}}\right) \dot{w}$	$\left(\frac{\partial X}{\partial Q}\right) Q$	$\left(\frac{\partial X}{\partial \eta}\right) \eta$; $\left(\frac{\partial X}{\partial \dot{\eta}}\right) \dot{\eta}$	$\left(\frac{\partial X}{\partial v}\right) v$				
ΔZ	$\frac{\partial Z}{\partial u} u$	$\frac{\partial Z}{\partial w} w$; $\frac{\partial Z}{\partial \dot{w}} \dot{w}$	$\left(\frac{\partial Z}{\partial Q}\right) Q$	$\frac{\partial Z}{\partial \eta} \eta$; $\frac{\partial Z}{\partial \dot{\eta}} \dot{\eta}$					
ΔM	$\left(\frac{\partial M}{\partial u}\right) u$	$\left(\frac{\partial M}{\partial w}\right) w$; $\left(\frac{\partial M}{\partial \dot{w}}\right) \dot{w}$	$\left(\frac{\partial M}{\partial Q}\right) Q$; $\frac{\partial M}{\partial \dot{Q}} \dot{Q}$	$\left(\frac{\partial M}{\partial \eta}\right) \eta$; $\frac{\partial M}{\partial \dot{\eta}} \dot{\eta}$	$\frac{\partial M}{\partial v} v$				
ΔY					$\left(\frac{\partial Y}{\partial v}\right) v$; $\left(\frac{\partial Y}{\partial \dot{v}}\right) \dot{v}$	$\left(\frac{\partial Y}{\partial P}\right) P$	$\frac{\partial Y}{\partial R} R$	$\left(\frac{\partial Y}{\partial \xi}\right) \xi$	$\left(\frac{\partial Y}{\partial \zeta}\right) \zeta$; $\left(\frac{\partial Y}{\partial \dot{\zeta}}\right) \dot{\zeta}$
ΔL					$\left(\frac{\partial L}{\partial v}\right) v$; $\left(\frac{\partial L}{\partial \dot{v}}\right) \dot{v}$	$\left(\frac{\partial L}{\partial P}\right) P$	$\left(\frac{\partial L}{\partial R}\right) R$	$\left(\frac{\partial L}{\partial \xi}\right) \xi$; $\left(\frac{\partial L}{\partial \dot{\xi}}\right) \dot{\xi}$	$\left(\frac{\partial L}{\partial \zeta}\right) \zeta$; $\left(\frac{\partial L}{\partial \dot{\zeta}}\right) \dot{\zeta}$
ΔN					$\left(\frac{\partial N}{\partial v}\right) v$; $\left(\frac{\partial N}{\partial \dot{v}}\right) \dot{v}$	$\left(\frac{\partial N}{\partial P}\right) P$	$\left(\frac{\partial N}{\partial R}\right) R$	$\left(\frac{\partial N}{\partial \xi}\right) \xi$	$\left(\frac{\partial N}{\partial \zeta}\right) \zeta$; $\left(\frac{\partial N}{\partial \dot{\zeta}}\right) \dot{\zeta}$

TABLE 2-1 KNOWN AERODYNAMIC STABILITY DERIVATIVES

Blanks indicate derivatives for which no estimate can be found.

prove quite accurate. Others are very poor and empirical methods or model testing are used. Even these fail at times and "educated guesses" from previous airplanes are used as a basis of determining stability derivatives for a new design. Tail effects are the most notable in this last category.

In all aerodynamics problems, considerable effort is made to non-dimensionalize the equations. The advantages of doing this are of considerable merit. For the most part, it is rather simply done. It has been found in both theory and actual practice that

$$F = \tfrac{1}{2}\rho V^2 S C_F \qquad \text{II-4}$$

Where S is a reference area, usually the wing area, V is the absolute reference velocity and C_F is some non dimensional coefficient. C_F varies with Mach Number and Reynolds Number depending on the force considered. (lift, drag, etc.) ρ is the density of the fluid through which the body is passing. Thus a term such as Zw has the following relationship:

$$Zw' = \frac{\partial Z}{\partial w} = \tfrac{1}{2}\rho V^2 S \left(\frac{\partial C_Z}{\partial w}\right) = \tfrac{1}{2}\rho V^2 S C_{Z_w} \qquad \text{II-5}$$

Here again V and S are the velocity and wing area terms respectively. The same concept applies to the moment terms except that a moment arm term must also appear in the equation. A generalized example is:

$$M = \rho V^2 S l C_m \qquad \text{II-6}$$

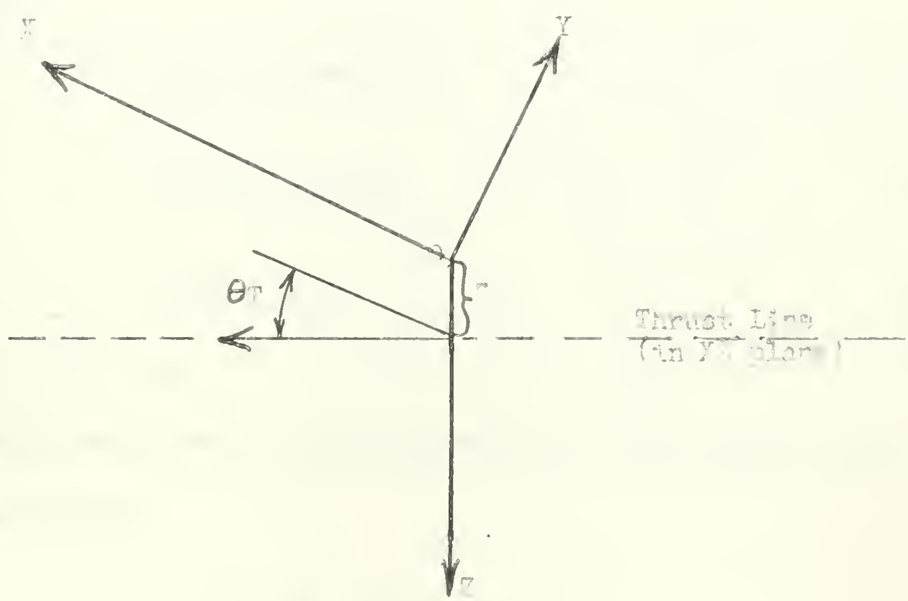
In this example M is some moment and C_M is a needed dimensionless coefficient to satisfy the equation. ρ, V and S are the same as

mentioned above while ℓ is the moment arm.

Likewise in a stability moment the derivative can be of the following form.

$$M_h = \rho V^2 S l \left(\frac{\partial C_m}{\partial h}\right) h = \rho V^2 S l C_{m_h} \qquad [illegible]$$

Another external force to be considered is that of thrust. This becomes an important factor in most flight conditions. The location of the thrust axis is seldom coincident with any reference axis in the body.



[illegible]

Assuming the thrust line lies in the xz plane, then the thrust forces can be resolved in this generalized manner.

$$X_T = T \cos \Theta_T \qquad \text{II-8}$$

$$Z_T = T \sin \Theta_T \qquad \text{II-9}$$

$$M_T = T_z \qquad \text{II-10}$$

For a steady state condition the equations become

$$X_{o_T} = To \cos \Theta_T \qquad \text{II-11}$$

$$Z_{o_T} = To \sin \Theta_T \qquad \text{II-12}$$

$$M_{o_T} = T_{oz} \qquad \text{II-13}$$

while in a disturbed condition where the body axis remains fixed in the mass,

$$X_T = T_1 \cos \Theta_T \qquad \text{II-14}$$

$$Z_T = T_1 \sin \Theta_T \qquad \text{II-15}$$

$$M_T = T_{1z} \qquad \text{II-16}$$

and $T_1 = To + \Delta T$ II-17

Assuming the air density remains constant so the engine thrust will not be affected

$$T = \left(\frac{\partial T}{\partial u}\right) u + \left(\frac{\partial T}{\partial \delta_{RPM}}\right) \delta_{RPM} \qquad \text{II-18}$$

or

$$T = T_u u + T_{\delta_{RPM}} \delta_{RPM}$$

These are the considerations for the external forces on a body in motion. While the dimensionless coefficient has been used extensively by the aeronautical engineering field, it will not be used further in this work. The primary mission of this work is to investigate the control response and paths for possible compensation.

The generalized equations of motion then appear:

$$m(\dot{U} + QW - RV) + mg\sin\Theta_0(\cos\Theta\cos\Psi) - mg(\cos\Theta_0\sin\phi_0)(\cos\Theta\sin\Psi) + mg(\cos\Theta_0\cos\phi_0)(\sin\Theta) = X_0 + X'_u u + X'_w w + X'_{\dot{w}}\dot{w} + X'_q q + X'_\eta \eta + X'_{\dot{\eta}}\dot{\eta} + X'_{\dot{q}}\dot{q} + T_0\cos\Theta_T + T_u\cos\Theta_T u + T_{\delta rpm}\cos\Theta_T\delta rpm \qquad \text{II-19}$$

$$m(\dot{V} + RU - PW) + mg\sin\Theta_0(\cos\Psi\sin\Theta\sin\phi - \sin\Psi\cos\phi) - mg(\cos\Theta_0\sin\phi_0)(\cos\Psi\cos\phi + \sin\Psi\sin\Theta\sin\phi) - mg(\cos\Theta_0\cos\phi_0)(\cos\Theta\sin\phi) = Y_0 + Y'_v v + Y'_{\dot{v}}\dot{v} + Y'_p p + Y'_r r + Y'_\xi \xi + Y'_\zeta \zeta + Y'_{\dot{\zeta}}\dot{\zeta} \qquad \text{II-20}$$

$$m(\dot{W} - QU + PV) + mg\sin\Theta_0(\cos\Psi\sin\Theta\cos\phi + \sin\Psi\sin\phi) - mg(\cos\Theta_0\sin\phi_0)(\sin\Psi\sin\Theta\cos\phi - \cos\Psi\sin\phi) - mg(\cos\Theta_0\cos\phi_0)(\cos\Theta\cos\phi) = Z_0 + Z'_u u + Z'_w w + Z'_{\dot{w}}\dot{w} + Z'_q q + Z'_\eta \eta + Z'_{\dot{\eta}}\dot{\eta} - T_0\sin\Theta_T - T_u\sin\Theta_T u - T_{\delta rpm}\sin\Theta_T\delta rpm \qquad \text{II-21}$$

$$A\dot{P} + \dot{K}_1 + QR(C-B) - E(PQ + \dot{R}) + cK_3 Q - bK_2 R = L_0 + L'_v v + L'_{\dot{v}}\dot{v} + L'_p p + L'_r r + L'_\xi \xi + L'_{\dot{\xi}}\dot{\xi} + L'_\zeta \zeta + L'_{\dot{\zeta}}\dot{\zeta} \qquad \text{II-22}$$

$$B\dot{Q} + \dot{K}_2 + PR(A-C) + E(P^2 - R^2) + aK_1 R - cK_3 P = M_0 + M'_u u + M'_w w + M'_{\dot{w}}\dot{w} + M'_q q + M'_{\dot{q}}\dot{q} + M'_\eta \eta + M'_{\dot{\eta}}\dot{\eta} + T_0 z + T_u uz + T_{\delta rpm}\delta rpm z \qquad \text{II-23}$$

$$C\dot{R} + \dot{K}_3 + PQ(B-A) + E(QR - \dot{P}) + bK_2 P - aK_1 Q = N_0 + N'_v v + N'_p p + N'_r r + N'_\xi \xi + N'_\zeta \zeta + N'_{\dot{\zeta}}\dot{\zeta} \qquad \text{II-24}$$

Thus, it has been shown that the equations of motion have the above external forces which will affect the rigid mass in motion. The assumptions made are:

1. The earth is assumed to be fixed in space and the Earth's atmosphere is assumed to be fixed with respect to the Earth.
2. The mass is a rigid body.
3. The mass of the body is assumed constant during the time of the problem.
4. For an aircraft or missile, the xz plane is assumed to have mirror symmetry.

These assumptions were made in the first section. In addition the following assumptions and simplifications are made in considering the aerodynamic forces.

1. The flow is assumed to be quasi-steady.
2. The effects of auxiliary aerodynamic equipment such as flaps, speed brakes, slats and spoilers are neglected.
3. The thrust can be affected only by change of speed and/or revolutions per minute. No change is considered for change in fuel mixture, manifold pressure, side slip or propeller fin effect.

These assumptions were made for several reasons. In the first and third cases, absence of adequate information on many of the coefficients makes ignoring or setting them equal to zero mandatory. In all probability

these coefficients are very small. Assumption two covers special cases. While these items may have considerable effect when used, they are eliminated from the general case at this time because an aircraft probably uses them less than two percent of its flying time. For the special cases, these effects may be quickly added.

The work in these first two sections is developed primarily for background purposes. There is nothing new or startling in this. For further examination of this subject, the reader is referred to the works in Ref. 7 and 8 which closely parallel the development in this work.

The aerodynamic force and moment terms carry a primed notation for ease in publication. In later sections these terms will be divided by mass or moment of inertia. These terms after division will not carry the prime (') notation. There are far more of the latter terms used. Thus, it is for ease of publication only and does not affect theory in any way.

3. The Airframe Equations in a Standard Form

This section will take the general equations developed in the previous chapters and simplify them. The simplification will extend to an aircraft in straight and level flight and subject to small perturbations. This is the standard method of dealing with the problem as found in present day texts. However, these equations will be used in block diagram form and in determinental arrays as developed in Chu's work. (1) The results will be compared with present day classical methods to determine validity of this determinantal concept.

The first simplification is the concept of the lateral system being independent of the longitudinal system. Therefore, the six equations of motion are divided into the two groups. The longitudinal group includes the directional motion in the X and Z directions and rotary motion about Y axis. The lateral group consists of directional motion along the Y axis and rotary motion about the X and Z axis. The axis system referred to here is the body axis system of the airplane.

Secondly, the aircraft is initially in a steady state condition. This is the reference state as mentioned previously. When the disturbance terms are all zero Equation II-19 reduces to:

$$m(\dot{U}_o + Q_o W_o - R_o V_o) + mg \sin \Theta_o = X_o + T_o \cos \Theta_T \qquad \text{III-1}$$

However, it was stated that this situation was restricted to longitudinal or lateral systems. Thus for motion along the X body axis, a yaw motion Ro and a side motion Vo would be zero. Further it was

stated that the aircraft was in a steady state reference condition.

Therefore $\dot{u}$ would be zero. Thus equation III-1 reduces to

$$m\,(QoWo) + mg\sin\Theta_0 = X'o = To\cos\Theta_T \qquad \text{III-2}$$

Similarly, it may be observed that equation II-20 for the steady state reduces to

$$Y'o = 0 \qquad \text{III-3}$$

Likewise equation II-21 reduces to

$$m\,(QoVo) = Z'_0 + mg\cos\Theta_0 + To\sin\Theta_T = 0 \qquad \text{III-4}$$

The moment equations all reduce to zero provided the rotor effects are neglected or considered zero.

$$Lo = Mo = No = 0 \qquad \text{III-5}$$

Recalling that only small disturbances were to be considered and these disturbances were to be in only the symmetrical or longitudinal system for the longitudinal group the derivative due to asymmetric variables of V, P, R, ξ and ζ are zero. In a similar manner, when the asymmetric system effects are considered the symmetric disturbances are ignored. This leaves the variables V, W, Q, and η out of the asymmetric equations. Finally, because of the small disturbances, the change angle Θ will be small so the approximation $\cos\Theta \cong 1$ and $\sin\Theta \cong \Theta$ is valid.

The result of the previous discussion is subtracting equations III-2 from II-19, eliminating the asymmetric changes and supplying the small disturbance considerations. The result is:

$$m\,(\dot{u} + \cancel{QoW} + \cancel{WoQ} + \cancel{WQ}) + mg\,\Theta\cos\Theta_0 = X_u'U + X_w'W + X_{\dot{w}}'\dot{W} + X_q'Q + X_\eta'\eta + X_{\dot{\eta}}'\dot{\eta} + Tu'\cos\Theta_T U + T_\delta'_{RPM}\cos\Theta_T'\delta RPM \qquad \text{III-6}$$

For the z direction taking III-4 from II-21 and using the same assumptions:

$$[illegible] \quad \text{III-7}$$

$$[illegible] + Z'_{\eta}\eta + Z'_{\dot\eta}\dot\eta = T_U U \sin\Theta_T = T_{\delta RPM}\delta_{RPM} \sin\Theta_T$$

For the moment in the y axis, the following applies:

$$I_y\dot{Q} = M_u'U + M_w'W + M_{\dot w}'\dot W + M_qQ + M_{\dot q}\dot Q + M'_{\eta}\eta + M'_{\dot\eta}\dot\eta + T_U U [illegible] + T_{\delta RPM}\,\delta_{RPM} \quad \text{III-8}$$

To complete the case for the symmetrical group, the action of the elevators must be included. From appendix I the elevator hinge moment effects are reduced to:

$$I_e\dot\eta = H_{e\eta}'\eta + H_{e\dot\eta}'\dot\eta + H_{ew}'W + H_{e\dot w}'\dot W + H_{eu}'U + H_{eq}'Q + \Delta F_e \quad \text{III-9}$$

In most cases where moment of inertia terms are handled it is usually expedient to use an axis system which is the principal inertia axis. This means that the product of inertia terms are zero. For aerodynamic motion problems, this applies for some cases.[8] This case is not one of them. What is gained in simplification of the rotary motion equations by use of principal axis is lost in attempting to handle the force equations with the wind velocity considerations in sine and cosine terms. By orienting the X axis into the reference wind velocity the result is:

$$V_c^2 = U^2 + v^2 + w^2 \quad \text{III-10}$$

where $U = U_o + u$.

Therefore, in the steady state reference condition $V_c = U_o$ and $V_o = W_o = 0$. With the assumption that u, v and w were very small, their products and squares can be neglected.

and since U_o^2 is much greater than $2\,U_o u$, then $V_c \cong U_o \cong U$ and these terms are interchangeable in this section.

As previously stated, the reference situation is a steady state motion and includes first order effects only. The orientation of the axis U_o is a finite value while $V_o = W_o = 0$. Furthermore, the author is hard pressed to visualize an aircraft or missile tumbling along with a Q_o motion although P_o exists in spinning missiles and bullets. Further $X_w = X_q = Z_{\dot{w}} = X_\eta = X_{\dot{\eta}} = T_{\delta\,RPM} = 0$ as an assumption. These terms are usually very small.

The "standard" determinant form as stated in Chapter 3 of Chu[1] is:

$$
\begin{array}{cccccc}
1 + PF_1 & PF_{11} & PF_2 & PF_{21} & PF_3 & PF_{31} \\
- PF_1 & & 1 + PF_2 & PF_{22} & PF_3 & PF_{32} \\
- PF_{13} & & - PF_2 & & 1 + PF_3 & PF_{33}
\end{array}
$$

where the PF indicates a performance function. The system may have n nodes. This example contains only three nodes. The "standard" block diagram for this form is:

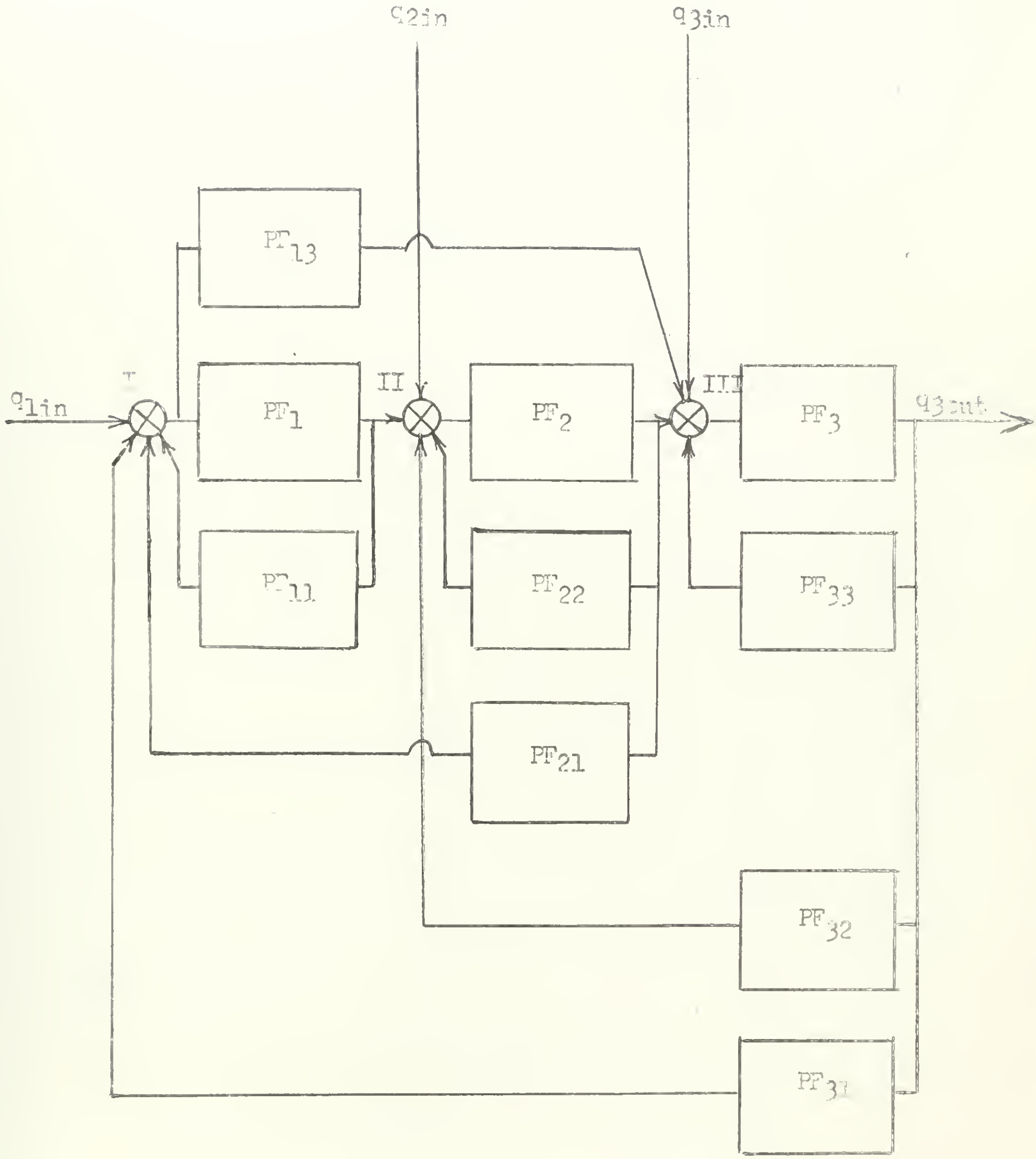


Figure 3-1 STANDARD BLOCK DIAGRAM DRAWN FROM 3x3 DETERMINANT

The nodes are labeled in Roman Numerals. The basic rules of nodal signals allow several inputs, but only one output. This output from the node goes directly to a performance function with a single subscript number. Signals may be picked off from the node output and fed forward to another node. An example of this is contained in the signal going through PF_{13}. In this case the signal is also multiplied by the function PF_{13} and then fed to node III. Feedback takes place in a similar manner. The signal output from the first system PF_1 is picked off and fed through PF_{11} back to node I. Similarly the other nodes function in the same manner.

For feedback blocks, the first subscript number denotes the output signal source for the feed back and the second subscript number the input node to which the feedback signal is going. The notable points about the determinant are:

1. The main diagonal always has terms of one plus the direct path performance function times the self-loop feedback performance function.

   $(1 + PF_1 PF_{11})$

2. All terms above and to the right of the main diagonal are feedback terms.

3. All terms below and to the left of the main diagonal of the "standard" determinant form are of a feed forward nature. All of these feed forward terms are negative. The term immediately below the main diagonal term is the

direct path performance function. Thus, where the second column main diagonal spot has $1 + PF_2 PF_{22}$, the direct path term PF_2 will appear in the location immediately below as $-PF_2$.

These rules are strictly mechanical. For a rigorous derivation of this system the reader is referred directly to Chu's work.

In Chapter 9 of Etkin [4], there exists a two degree of freedom situation where θ and W vary and the speed in the forward direction is constant. This is the short period approximation. The equations using the LaPlace transforms and nondimen ional coefficients are:

$$(\text{[illegible]}_{\mu} - \text{[illegible]}_{\alpha})\bar{\alpha} - (\text{[illegible]}_{\mu} + \text{[illegible]})\bar{\theta} - \text{[illegible]}_{\eta}\eta = 0 \qquad \text{III-11}$$

$$-\text{[illegible]}_{\alpha}\bar{\alpha} + (\text{[illegible]}_{\beta}\text{[illegible]} - \text{[illegible]}_{mq})\bar{\theta} - \text{[illegible]}_{m\eta}\eta = 0 \qquad \text{III-12}$$

$$-\text{[illegible]}_{h\alpha}\bar{\alpha} - \text{[illegible]}\,\theta + (\text{[illegible]}^2 - \text{[illegible]}_{h\eta})\eta = \Delta C_{f_e} \qquad \text{III-13}$$

This author also grouped his acceleration terms such that G_{h_η} included C_{η} and $C_{h\dot{\eta}}$. Similar situations exist for $G_{n\alpha}$, $G_{m\alpha}$, and G . Note that $r \approx \dot{\psi}$ and $p \approx \dot{\phi}$ for this consideration. From these equations a block diagram is evolved.

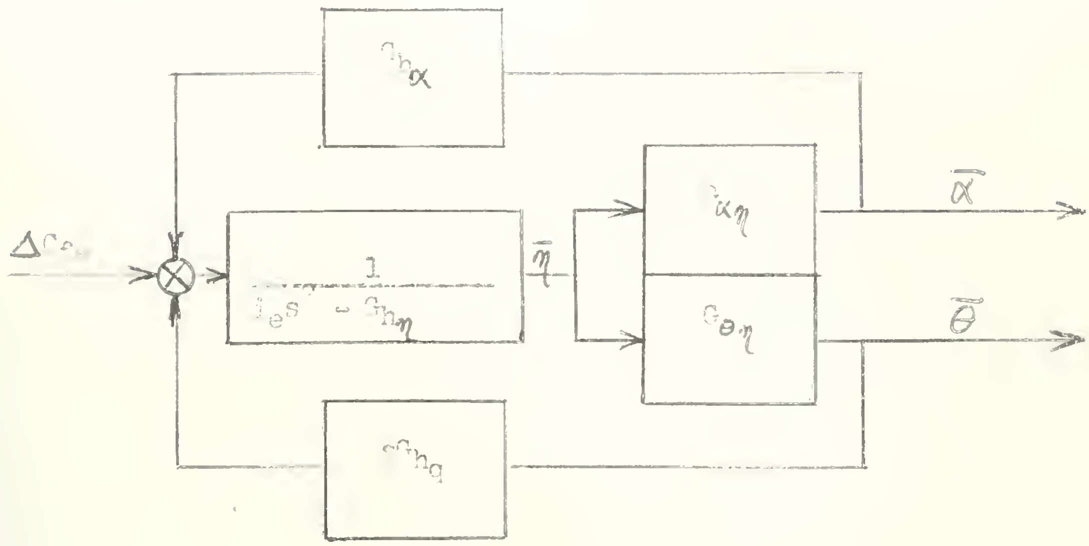


Figure 3-2 BLOCK DIAGRAM OF TWO DEGREES OF FREEDOM IN THE LONGITUDINAL SYSTEM SHOWN BY ETKIN[4]

Further, Etkin derives:

$$C_{\theta\eta} = \frac{\bar{\theta}}{\eta} = \frac{C_{m_\eta}(2\mu s - C_{z_\alpha}) + C_{z_\eta} C_{m_\alpha}}{(2\mu s - C_{z_\alpha})(i_B s^2 - C_{m_q} s) - C_{m}(2\mu + C_{z_q}) s} \qquad \text{III-14}$$

$$C_{\alpha\eta} = \frac{\Delta\alpha}{\eta} = \frac{C_{z_\eta}(i_B s^2 - C_{m_q} s) + C_{m}(2\mu + C_{z_q}) s}{(2\mu s - C_{z_\alpha})(i_B s^2 - C_{m_q} s) - C_{m_\alpha}(2\mu + C_{z_q}) s} \qquad \text{III-15}$$

Not it is intended to use Chu's work involving the use of "standard" block diagrams and "standard" determinant form to prove that the "standard" system applies for the airframe as well as a multiloop servo system.

The Z, M and η equations of motion with the applied assumptions and $\theta_o = 0$ are:

$$\dot{w} - Z_w w - Z_{\dot{w}} \dot{w} - Z_q q - Z_\eta \eta - Z_{\dot{\eta}} \dot{\eta} = 0 \qquad \text{III-16}$$

$$\dot{q} - M_w w - M_{\dot{w}} \dot{w} - M_q q - M_\eta \eta - M_{\dot{\eta}} \dot{\eta} = 0 \qquad \text{III-17}$$

$$\ddot{\eta} - 10_\eta \eta - 10_{\dot{\eta}} \dot{\eta} - 10_w w - 10_{\dot{w}} \dot{w} - 10_q q = \Delta Fe \qquad \text{III-18}$$

Recalling that $\omega = U_o \tan \Delta\alpha$ and for small angles

$\omega \cong U_o \Delta\alpha$ and $\dot{\theta} \cong q.$

The equations in LaPlace notation then become:

$$(s^2 - (s10_{\dot{\eta}} + 10_\eta))\eta - (s10_{\dot{w}} + 10_w) U_o \Delta\alpha - (s10_q)\theta = \Delta Fe \qquad \text{III-19}$$

$$(s^2 - sM_q)\theta - (sM_{\dot{\eta}} + M_\eta)\eta - (sM_{\dot{\alpha}} + M_\alpha) U_o \Delta\alpha = 0 \qquad \text{III-20}$$

$$(s - Z_w) U_o \Delta\alpha - (sU_o + sZ_q)\theta - (sZ_{\dot{\eta}} + Z_\eta)\eta = 0 \qquad \text{III-21}$$

These equations in standard determinant form now become:

η	$\Delta\alpha$	θ
$1 - \dfrac{(s10_{\dot{\eta}} + 10_\eta)}{s^2}$	$-\dfrac{(s10_{\dot{\alpha}} + 10_\alpha)}{s^2}$	$-\dfrac{s10_q}{s^2}$
$-\dfrac{(sZ_{\dot{\eta}} + Z_\eta)}{s}$	$1 - \dfrac{Z_\alpha}{s}$	$-\dfrac{s(U_o + Z_q)}{s}$
$-\dfrac{(sM_{\dot{\eta}} + M_\eta)}{s^2}$	$-\dfrac{(sM_{\dot{\alpha}} + M_\alpha)}{s^2}$	$1 - \dfrac{sM_q}{s^2}$

Ordinarily, it is not necessary to divide by s or s^2. However, it is done here to demonstrate more closely the standard form.

To take the differential equations and arrange them by rows and columns to see the actual physical system is to no avail. The arrangement does not exist in the above form. It will be shown how to achieve this form later.

Consider the "standard form" array as set forth above. The block diagram for this in Chu's "standard form" is:

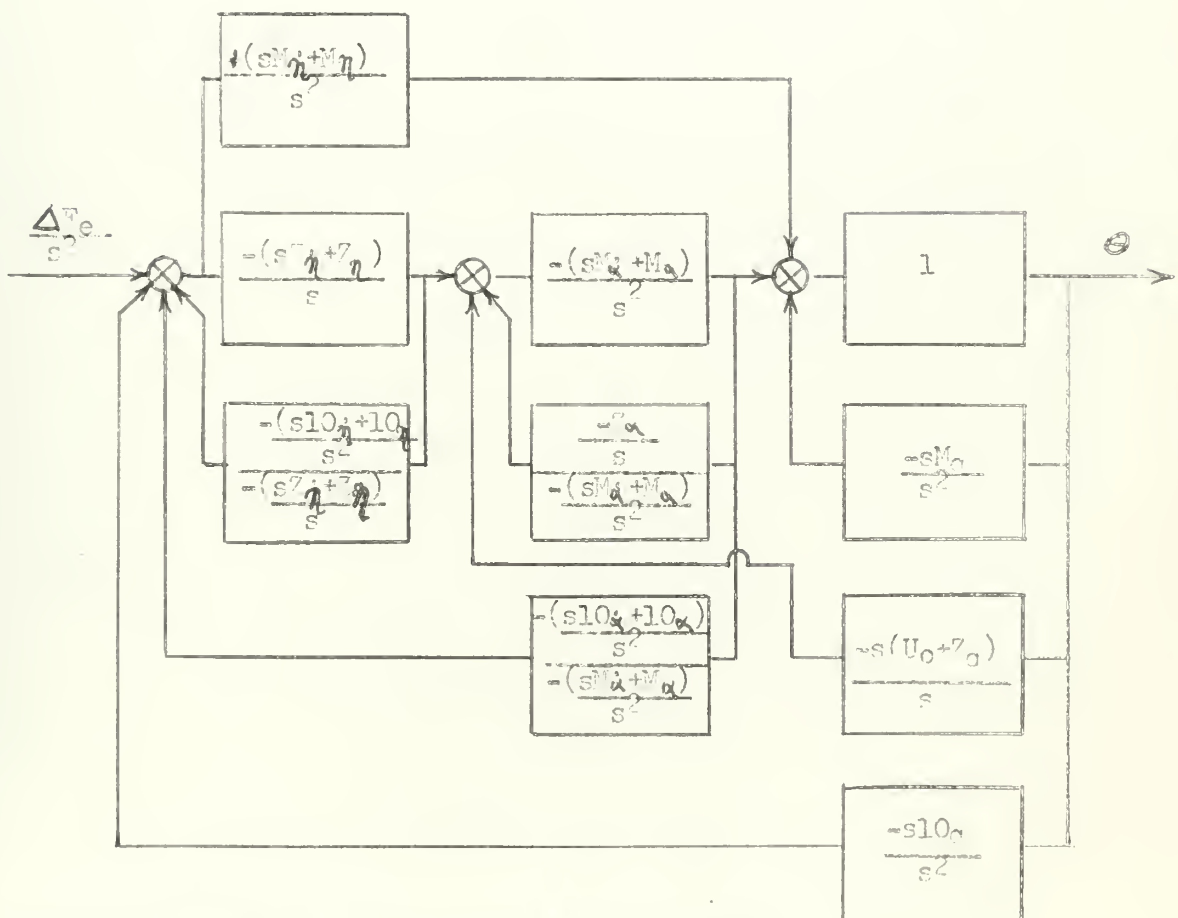


Figure 3-3 RESULT OF BLOCK DIAGRAMING EQUATIONS III-19,20 and 21

The above block diagram then is a three node system but the outputs from the first and second main path blocks are nothing that is readily recognized in an airframe. Because of the fact that the main block following the third node is 1, the output here is θ .

A few simple block diagram manipulations clears the situation and shows true signal outputs that are physically realized in the air-frame. Block diagram manipulations allow the changing of a pick off point provided the transfer function is corrected. Further, two blocks can be combined into one equivalent transfer function.

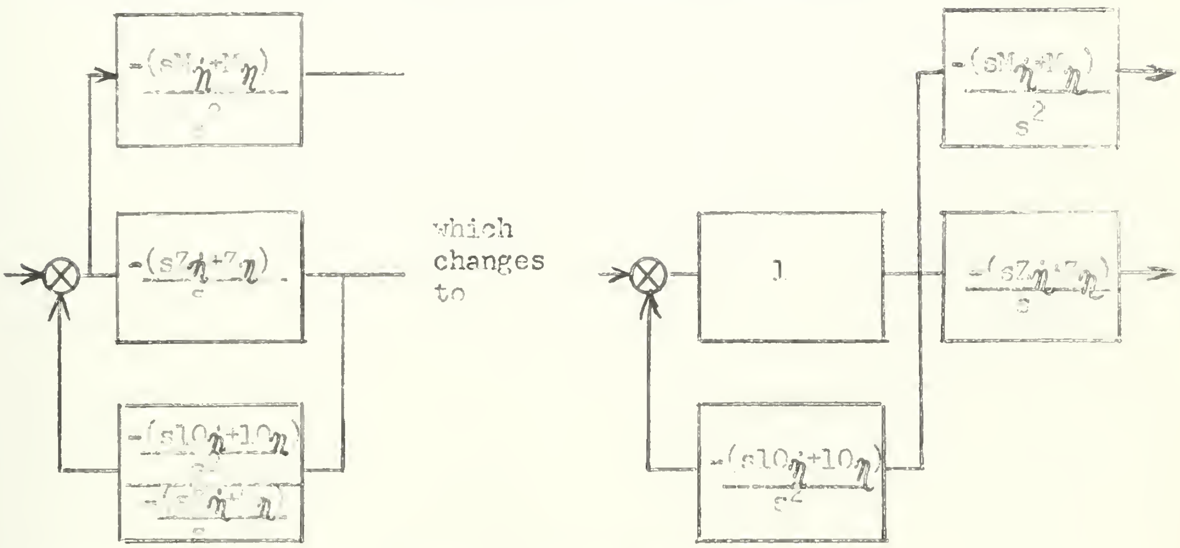


Figure 3-4 EQUIVALENT BLOCK DIAGRAM TERMS

Adding a dummy node this now changes to:

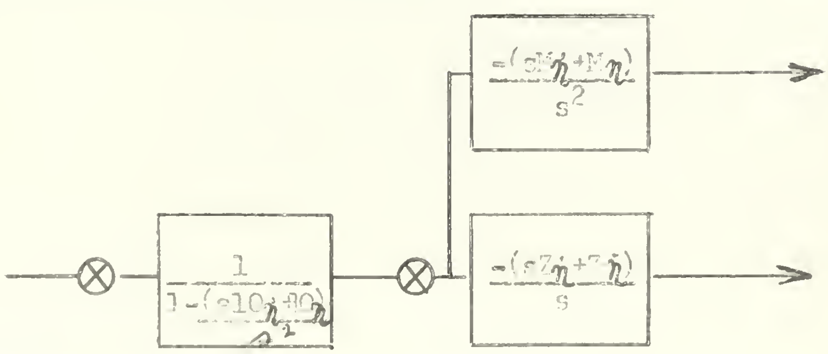


Figure 3-5 FURTHER BLOCK DIAGRAM COMBINATIONS

The result of using these two maneuvers rearranges the block diagram to the following:

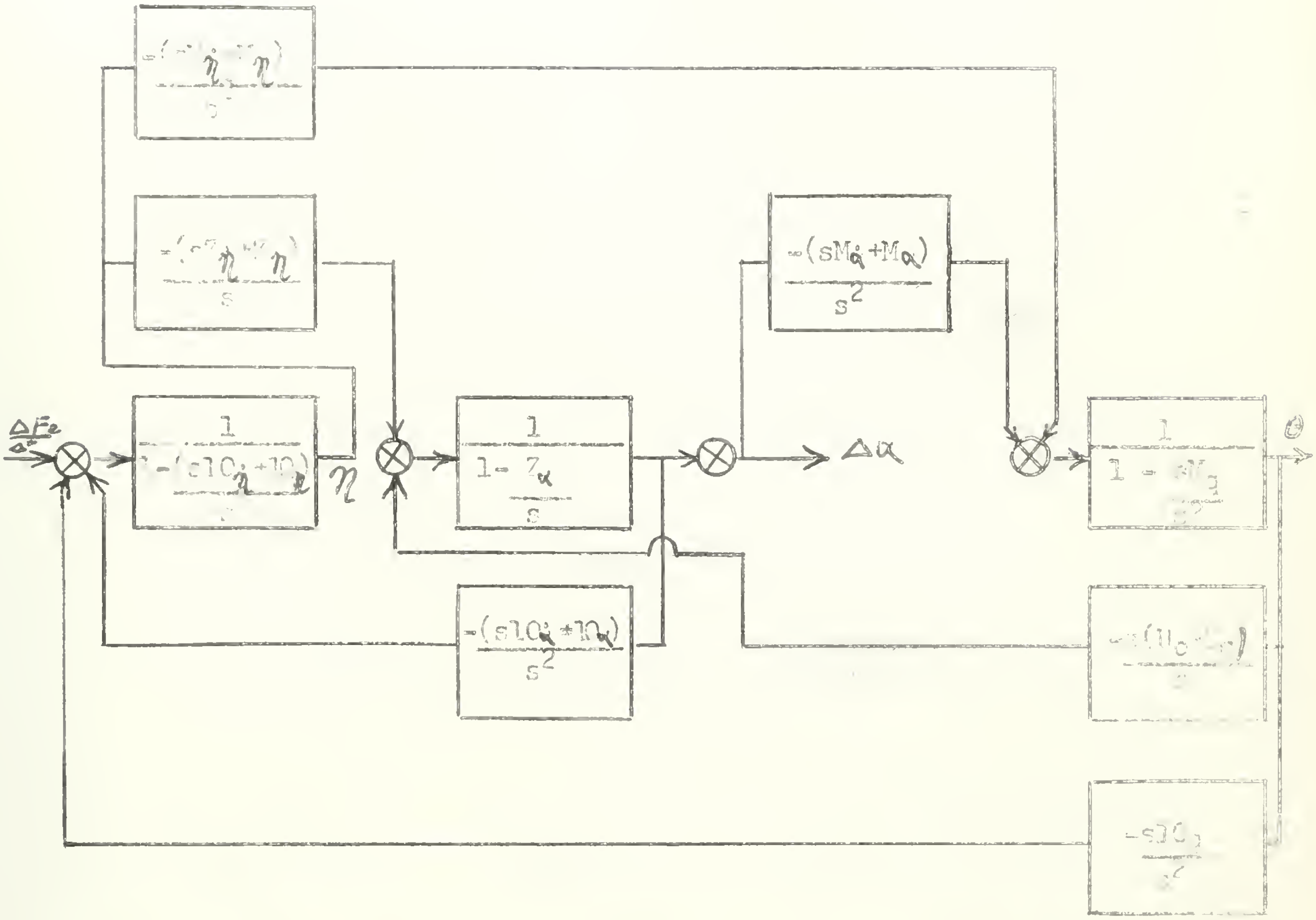


Figure 3-6 REARRANGED BLOCK DIAGRAM OF FIGURE 3-3, USING MANIPULATIONS OF FIGURES 3-4 AND 3-5

Maneuvering the above diagram location, the block diagram takes the form of Fig. 3-7, which coincides with Fig. 3-2.

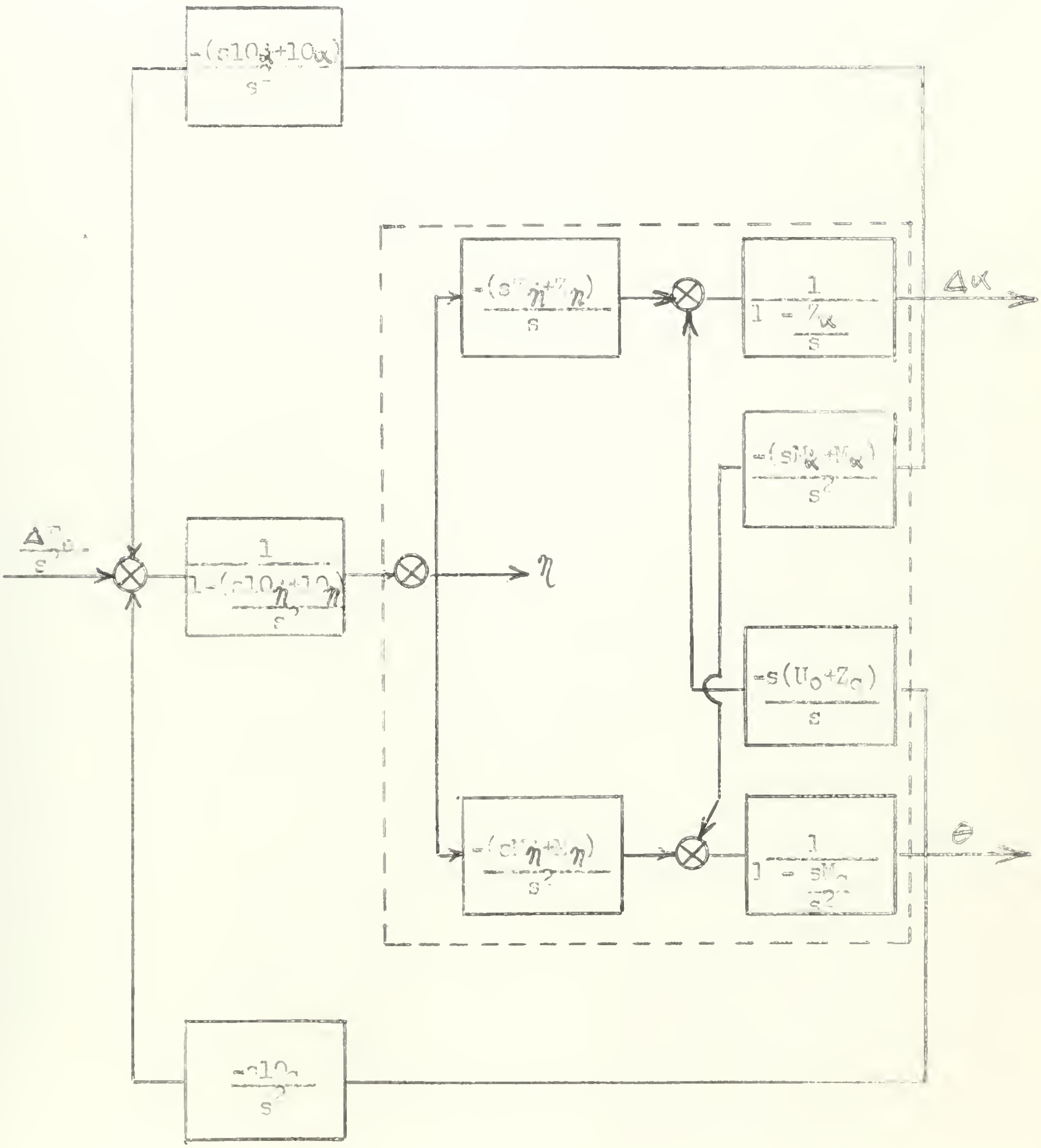


Figure 3-7 TWO DEGREES OF FREEDOM IN AN AIRFRAME LONGITUDINAL SYSTEM
(Rearranged Diagram of Figure 3-6 to compare with Figure 3-[illegible])

It is noted that this block diagram approaches the form set down in Etkin except for nondimensionalizing the terms. The dashed area in the diagram is what Etkin labels $G_{\theta\eta}$ and $G_{\alpha\eta}$.

The determinant from the above dashed portion of the block diagram is:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{(sZ_{\dot{\eta}} + Z_\eta)}{s} & 1 - \frac{Z_\alpha}{s} & 0 & -\frac{s(U_o + Z_q)}{s} & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -\frac{(sM_{\dot{\eta}} + M_\eta)}{s^2} & 0 & -\frac{(sM_{\dot{\alpha}} + M_\alpha)}{s^2} & 1 - \frac{sM_q}{s^2} & 0 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

Evaluating for the transfer functions $\frac{\theta}{\eta}$ and $\frac{\Delta\alpha}{\eta}$, the results will be in the form $\frac{G_{jk}\Delta_{iv}}{\Delta}$

This form is the type expressed by Chu[1] and Thaler[12]. The Δ term, which is the whole determinant, reduces to:

$$\Delta = \begin{vmatrix} 1 - \frac{Z_\alpha}{s} & 0 & -\frac{s(U_o + Z_q)}{s} \\ -1 & 1 & 0 \\ 0 & -\frac{(sM_{\dot{\alpha}} + M_\alpha)}{s^2} & 1 - \frac{sM_q}{s^2} \end{vmatrix}$$

which is $\Delta = \left(1 - \frac{Z_\alpha}{s}\right)\left(1 - \frac{sM_q}{s^2}\right) - \frac{s(U_o + Z_q)}{s}\frac{(sM_{\dot{\alpha}} + M_\alpha)}{s^2}$ III-22

or $\Delta = \frac{1}{s^3}\left[(s - Z_\alpha)(s^2 - sM_q) - s(U_o + Z_q)(sM_{\dot{\alpha}} + M_\alpha)\right]$

To consider the $\Delta\alpha$ term the cofactor Δ_y of this determinant is obtained by crossing through the first row and the second column.

$$-\Delta_1 = \begin{vmatrix} \frac{-(sZ_{\dot\eta} + Z_\eta)}{s} & 0 & \frac{-s(U_o + Z_q)}{s} \\ 0 & 1 & 0 \\ \frac{-(sM_{\dot\eta} + M_\eta)}{s^2} & \frac{-(sM_{\dot\alpha} + M_\alpha)}{s^2} & 1 - \frac{sM_q}{s^2} \end{vmatrix}$$

$$\Delta_{12} = \frac{+1}{s} \left[+ (sZ_{\dot\eta} + Z_\eta)(s^2 - sM_q) + s(U_o + Z_q)(sM_{\dot\eta} + M_\eta) \right] \qquad \text{III-23}$$

The main feed forward transfer function between the input signal from node 2 to the output is +1 in this case. This is also true of node 4 input to output. Hence $G_{2\Delta\alpha} = +1$.

The transfer function

$$G_{\Delta\alpha\eta} = \frac{\Delta\alpha}{\eta} = \frac{G_{2\Delta\alpha}\ \Delta_{12}}{\Delta}$$

$$G_{\Delta\alpha\eta} = \frac{(sZ_{\dot\eta} + Z_\eta)(s^2 - sM_q) + s(Uo + Z_q)(sM_{\dot\eta} + M_\eta)}{(s - Z_\alpha)(s^2 - sM_q) - s(U_o + Z_q)(sM_{\dot\alpha} + M_\alpha)} \qquad \text{III-24}$$

It can be noted that this equation III-24 is the same equation as III-15 given by Etkin save the non-dimensionalizing terms.

To evaluate the transfer function Θ / η , the same method is used

$C_\Theta = +1$

$$\Delta_{14} = \begin{vmatrix} \dfrac{-(sZ_{\dot\eta} + Z_\eta)}{s} & 1 - \dfrac{Z_\alpha}{s} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \dfrac{-(sM_{\dot\eta} + M_\eta)}{s^2} & 0 & \dfrac{-(sM_{\dot\alpha} + M_\alpha)}{s^2} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_{14} = \begin{vmatrix} \dfrac{-(sZ_{\dot\eta} + Z_\eta)}{s} & 1 - \dfrac{Z_\alpha}{s} & 0 \\ 0 & -1 & 1 \\ \dfrac{-(sM_{\dot\eta} + M_\eta)}{s^2} & 0 & \dfrac{-(sM_{\dot\alpha} + M_\alpha)}{s^2} \end{vmatrix}$$

$$\Delta_{1k} = \frac{1}{\ldots}\left[(sM_{\dot\eta} + M_\eta)(s - Z_\alpha) + (sM_{\dot\alpha} + M_\alpha)(sZ_{\dot\eta} + Z_\eta)\right] \qquad \text{III-25}$$

Therefore:

$$G_{\theta\eta} = \frac{\theta}{\eta} = \frac{G_{1\theta}\,\Delta_{1k}}{\Delta}$$

$$G_{\theta\eta} = \frac{(sM_{\dot\eta} + M_\eta)(s - Z_\alpha) + (sM_{\dot\alpha} + M_\alpha)(sZ_{\dot\eta} + Z_\eta)}{(s - Z_\alpha)(s^2 - sM_q) - s(M_q + Z_q)(sM_{\dot\alpha} + M_\alpha)} \qquad \text{III-26}$$

Here again, excepting the non-dimensionalizing, the result is the same. Equation III-26 is equal to equation III-14 as given by Etkin.

The system of "standard" forms then works for the airframe differential equations after some manipulation. It can be noted that there is a method of making up this standard form without the necessity of the manipulation which will yield physically realizable output signals. When the block diagram of Fig. 3-6 is converted into a determinant for solution of the transfer function, there appears a series of alternating rows where the equation −1 + 1 = 0 appears. It also appears that these simple equations occur in a sequence where the plus one term is placed in the 2n! row and column. n is the number of differential equations in the problem. The minus one term is placed in the 2n! rows and (2n!−1) columns. All feedforward terms (those below and to the left of the main diagonal) are moved one column to the right and remain in the same row. This doubles the size of the determinant. However, there are several zero terms introduced and the actual work involved in

evaluation of the determinant is not markedly increased.

The next item is to look at the lateral motion problem. It is slightly more complex because there are five differential equations of motion instead of four.

Equation II-2 with simplifications becomes:

$$(\dot{v} + U_0 r) - \phi m g \cos\Theta_0 - Y_v V - Y_p - Y_R R - Y_\zeta \zeta = 0 \qquad \text{III-27}$$

$$(\dot{\beta} + r) - \frac{\phi g \cos\Theta_0}{U_0} - Y_\beta \beta - \frac{Y_p P}{U_0} - \frac{Y_r r}{U_0} - \frac{Y_{\zeta} \dot{\zeta}}{U_0} = 0 \qquad \text{III-28}$$

The moment equation II-22 becomes

$$\dot{P} - \frac{E}{A}\dot{R} - L_\beta \beta - L_p P - L_r r - L_\xi \xi - L_\zeta \zeta = 0 \qquad \text{III-29}$$

and equation II-24 becomes

$$\dot{R} + \frac{E}{C}\dot{P} - N_\beta \beta - N_p P - N_r R - N_\xi \xi - N_\zeta \zeta - N_{\dot{\zeta}} \dot{\zeta} = 0 \qquad \text{III-30}$$

The aileron hinge motion equations from Appendix I are

$$\ddot{\zeta} - 11_p p - 11_r r - 11_\xi \xi - 11_\zeta \zeta = \Delta F_a \qquad \text{III-31}$$

And also from Appendix I, the rudder hinge motion is

$$\ddot{\zeta} - 12_\beta \beta - 12_p p - 12_r r - 12_\zeta \zeta - 12_{\dot{\zeta}} \dot{\zeta} = \Delta F_r \qquad \text{III-32}$$

From these terms a determinantal array can be made. It should be noted that this time a set of equations of $-1 + 1 = 0$ will be inserted in the system. This permits direct drawing of the block diagram and omitting the need of manipulation. Table 3-1 then presents this array From this Fig. 3-8 is drawn directly.

	ζ		ψ		β		ϕ		ξ
$1-\frac{(sl2_{\dot\zeta}+l2_{\zeta})}{s^2}$	0	$\frac{-sl2_r}{s^2}$	0	$\frac{-l2_\beta}{s^2}$	0	$\frac{-l2_p s}{s^2}$	0	0	0
-1	1	0	0	0	0	0	0	0	0
0	$\frac{-(sN_{\dot\zeta}+N_\zeta)}{s^2}$	$1-\frac{sN_r}{s^2}$	0	$\frac{-N_\beta}{s^2}$	0	$\frac{s(\frac{E}{C}-N_p)}{s^2}$	0	$\frac{-sN_\xi}{s^2}$	0
0	0	-1	1	0	0	0	0	0	0
0	$\frac{-Y_\zeta}{s}$	0	$\frac{-s(1-Y_r)}{s}$	$1-\frac{Y_\beta}{s}$	0	$\frac{-s(g+Y_p)}{s}$	0	0	0
0	0	0	0	-1	1	0	0	0	0
0	$\frac{-L_\zeta}{s^2}$	0	$\frac{-s(\frac{E}{A}+L_r)}{s^2}$	0	$\frac{-L_\beta}{s}$	$1-\frac{sL}{s^2}$	0	$\frac{-L_\xi}{s^2}$	0
0	0	0	0	0	0	-1	1	0	0
0	0	0	$\frac{-sl1_r}{s^2}$	0	0	0	$\frac{-sl1_p}{s^2}$	$1-\frac{(sl1_{\dot\xi}+l1_\xi)}{s^2}$	0
0	0	0	0	0	0	0	0	-1	1

TABLE 3-I DETERMINANT ARRAY OF LATERAL SYSTEM EQUATIONS FOR SMALL DISTRUBANCES

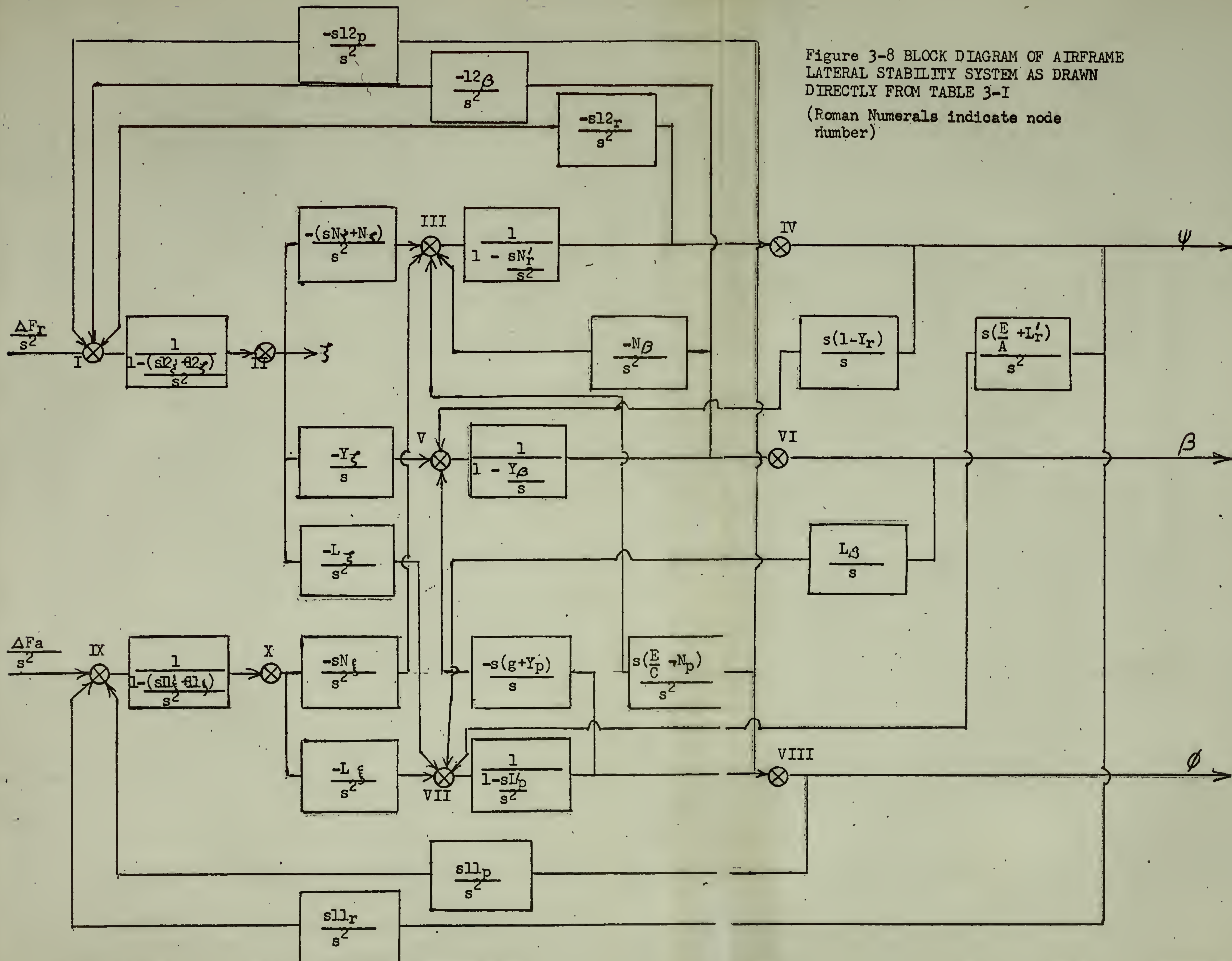


Figure 3-8 BLOCK DIAGRAM OF AIRFRAME LATERAL STABILITY SYSTEM AS DRAWN DIRECTLY FROM TABLE 3-I

(Roman Numerals indicate node number)

The resulting block diagram very clearly parallels that of the lateral system shown by Etkin in Chapter 9[4]. The "standard" form to achieve this block diagram differs.

There is, then, the question of why does this differ from the original "standard" form set forth by Chu.

Chapter Two of Chu[1] discusses at some length the idea of arrangement of varying performance functions into a "standard" block diagram. The individual pieces of hardware are at hand with known individual performance functions. The orderly arrangement of these permit the construction of a "standard" determinant for the whole system. From this, a multiloop servo system can be handled in generalized results. In the case of the airframe, the determinant formed by the differential equations was known and the block diagram had to be formed. Admittedly a block diagram (Fig. 3-3) was made up but the output signals are not the physically realizable type that are generally known to pilots and engineers. The same performance function results from the system. Both are mathematically correct.

The difference lies in the fact that Chu is taking individual pieces of hardware where the functions are known. The blocks are made up and then the "standard" determinant form is made. In the case of the airplane, the equations of motion are known and a determinant is immediately formed. Each column contains the performance functions multiplied by the direct function signal output term. Chu's standard form dictates that all terms above the main diagonal are multiplied by

the node output or input to the direct function. Steps must be taken then to change this in the differential equations. The addition of the alternating $-1 + 1 = 0$ equations solves this very nicely and in no way alters the value of the equation.

A series of evaluations of determinant forms is given in Appendix III. The results show that the addition of these terms in no way changes the final value of a determinant or a cofactor value.

Referring to Fig. 3-6, there are other block forms that may change the values within the blocks, but will not alter the final outcome. Consider the first node in Fig. 3-6 as an example. The four forms expressed in Fig. 3-9 are equivalent. The original form is shown in Fig. 3-9(a). The self-loop feedback path is combined with the direct path in Fig. 3-9(b). The s^2 term is eliminated and the performance functions are then in the form shown in Fig. 3-9(c). A further change can be made by dividing the nodal input terms by the denominator of the direct path feed-forward function. The result is then in the form shown in Fig. 3-9(d). The performance functions in each block differ somewhat, but the final result remains the same. These manipulations allow the user more freedom in the forms of the performance functions. It allows changes in form to fit the needs of the particular problem.

The pick-off point for the feed forward terms does differ in requirements. It is seen in Figs. 3-9(a) and (d) that the pick-off can be either off the output signal of node I or II. The final result will not change. However, this is not the case in Figs. 3-9(b) and (c). The

pick-off point must be off the output signal of node II. The final result will differ if the output of node I was tapped in the latter case.

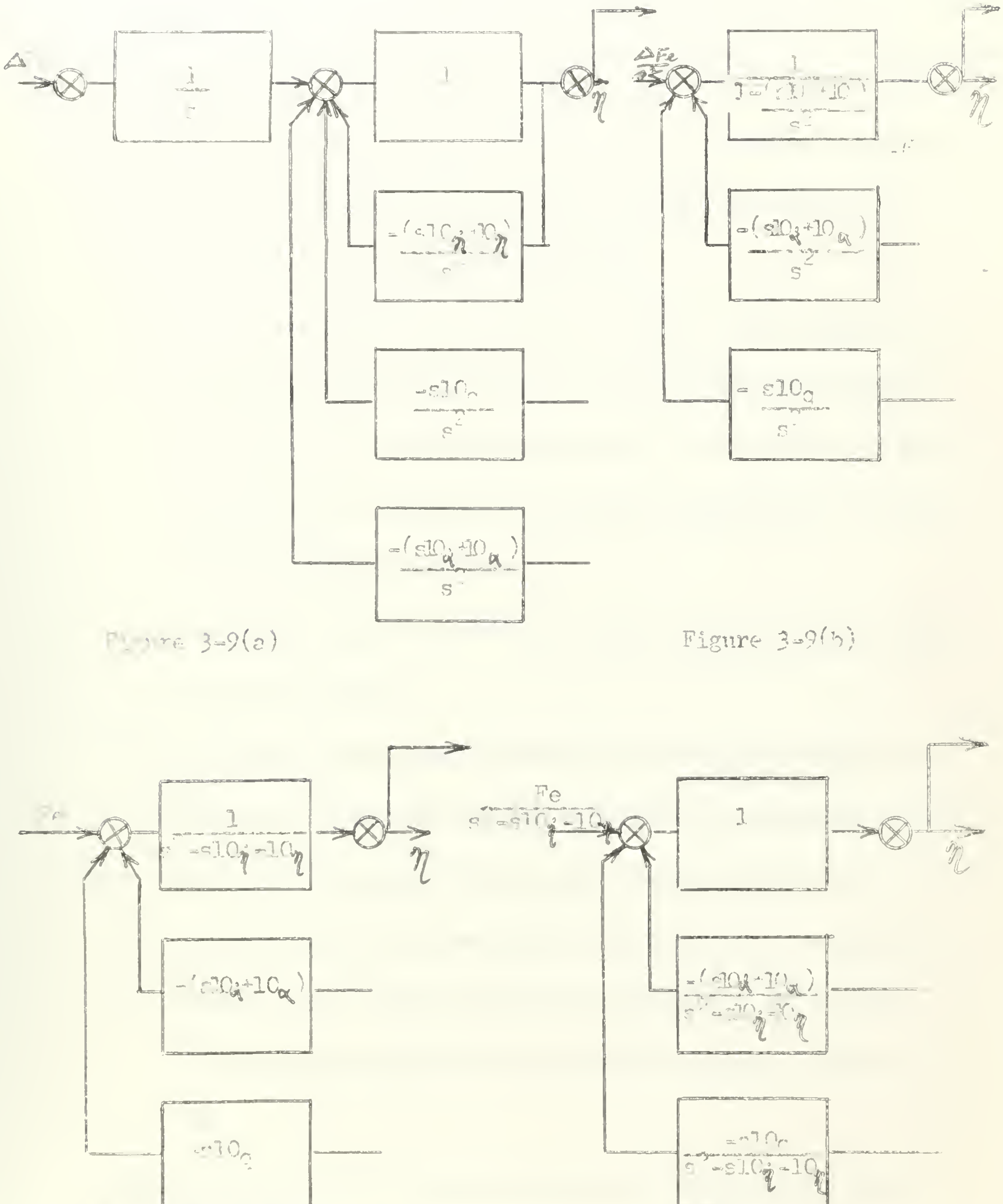


Figure 3-9(a) Figure 3-9(b)

Figure 3-9(c) Figure 3-9(d)

Figure 3-9 MULTIPLE BLOCK VALUE VARIATIONS AT A NODE

For converting a set of linear differential equations into the standard block diagram form, the following rules can be applied.

1. Between each differential equation, an equation $-1 + 1 = 0$ shall be inserted. An additional equation $-1 + 1 = 0$ shall be added at the bottom of the set so the number of additional equations shall equal the number of differential equations originally in the problem.
2. The arrangement of the values of these added equations in the determinant shall be such that the positive one value is always in the main diagonal block. The minus one term shall always be placed in the block immediately to the left of the main diagonal term.
3. All other blocks in the rows of these added equations shall be of zero value.
4. All finite performance functions of the original differential equations which are located below the main diagonal shall be moved one place to the right. The vacated blocks shall be replaced with a zero value. This last rule is optional to the user. It does save the possibility of error which could appear because of cases shown in Fig. 3-9(b) and (d).

With the simplified rules of making a standard block diagram from the standard determinant, a valuable tool is developed for many purposes. It can provide a student with a means to draw and visualize

a complex feedback system such as an airframe from a group of differential equations. For the research groups, it makes analog computer programming a simplified matter with respect to amplifier usage. In this latter case, pitfalls do exist. The airframe is no simple matter. The reader is referred to Howe[8] for an excellent coverage of the airframe problem for analog computers. Most important, though, is the fact that this standard form of Chu's is valid for any multi-loop linear feedback system whether it is electrical, mechanical or a combination of the two. This then puts a regular form to the multi-loop control system from which design or synthesis can proceed in an orderly manner.

4. The Control Problem for an Airframe

The previous three sections have discussed only the airframe itself. It is quite reasonable to say that while these point out many of the responses, they achieve little toward solving the control problem. In simplified terms, the control problem is resolved into the block diagram of Fig. 4-1.

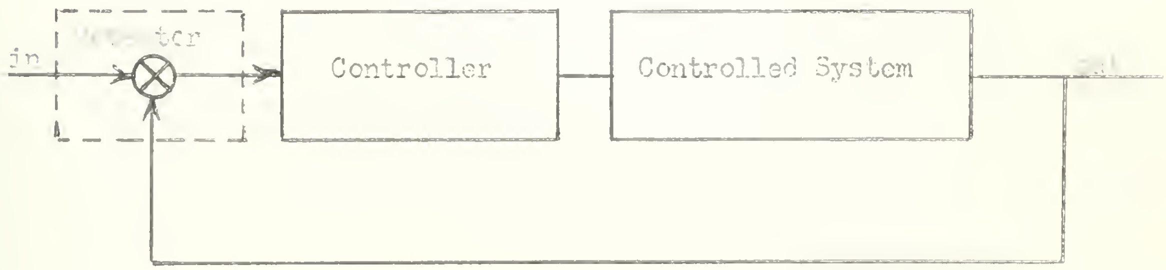


Figure 4-1 A SIMPLIFIED CONTROL SYSTEM

Indeed this diagram is an oversimplification of the aircraft control problem. However, it does point up the fact that the previous three chapters have dealt only with the controlled system block. Any complete study of the airframe control problem must include the other block components of the amplifier and the error detector. The problem is a multiloop type and of no small magnitude.

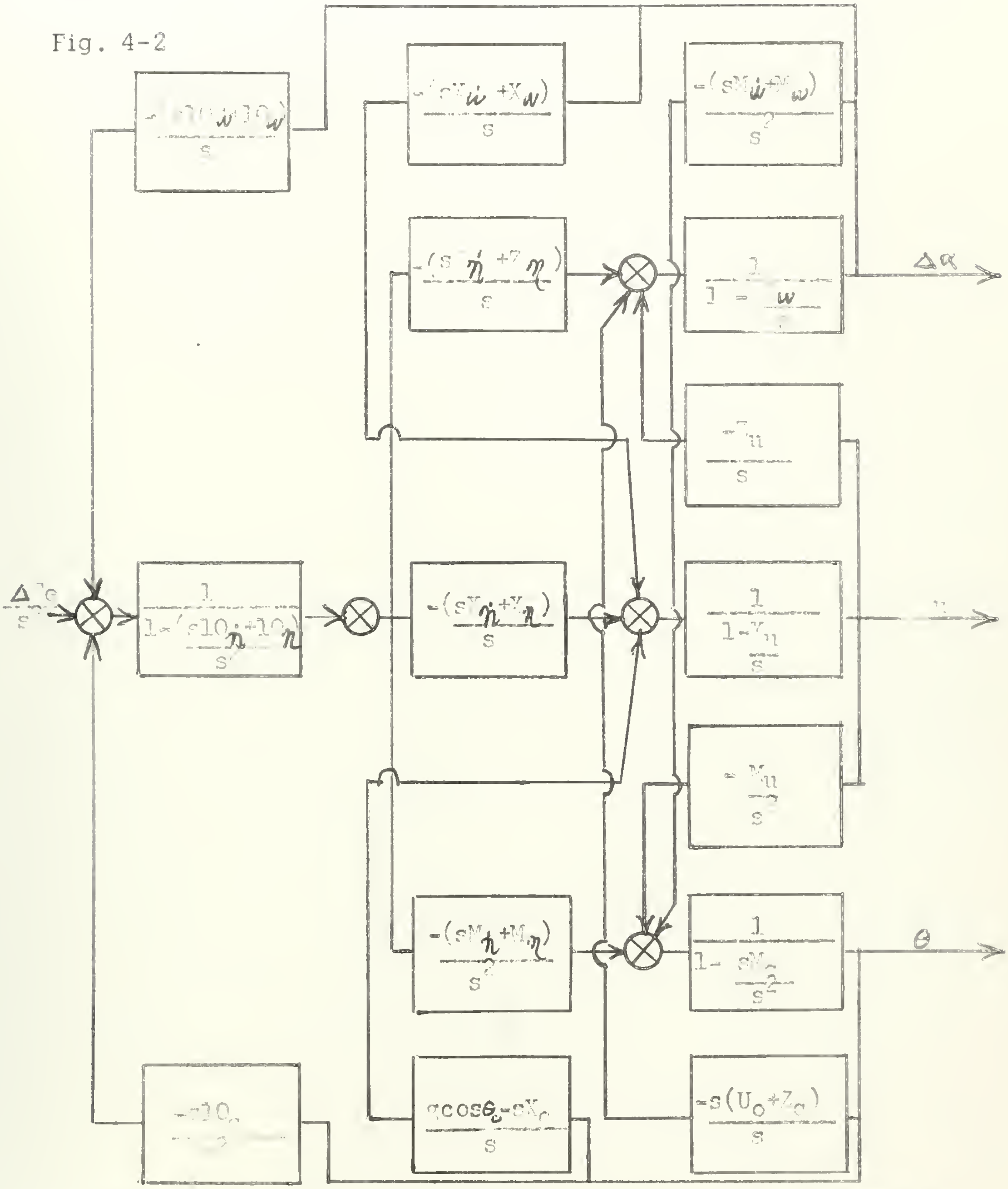
For the control problem, the diagramming becomes a simple extension of the block diagram developed in Section III. Consider the standard determinant for the longitudinal system.

The determinantal array for the longitudinal system is as follows

	η		u		$\Delta\alpha$		θ
$1-\frac{(sI_0\dot{\eta}+I_0\eta)}{s^2}$	0	0	0	$-\frac{(sI_0\dot{\alpha}+I\alpha)}{s^2}$	0	$-\frac{sI_0 q}{s^2}$	0
-1	1	0	0	0	0	0	0
$-\frac{(sX_{\dot{\eta}}+X_\eta)}{s}$	0	$1-\frac{X_u}{s}$	0	$-\frac{(sX_{\dot{w}}+X_w)}{s}$	0	$\frac{g\cos\theta-X_q}{s}$	0
0	0	-1	1	0	0	0	0
$-\frac{(sZ_{\dot{\eta}}+Z_\eta)}{s}$	0	$-\frac{Z_u}{s}$	0	$1-\frac{Z_\alpha}{s}$	0	$-\frac{s(V_0 Z_q)}{s}$	0
0	0	0	0	-1	1	0	0
$-\frac{s(M_{\dot{\eta}}+M_\eta)}{s^2}$	0	$-\frac{M_u}{s^2}$	0	0	$-\frac{s(M_{\dot{\alpha}}+M_\alpha)}{s^2}$	$1-\frac{sM_q}{s^2}$	0
0	0	0	0	0	0	-1	1

From this a block diagram can be made up from the determinantal array.

Fig. 4-2



LONGITUDINAL SYSTEM OF AIRFRAME WITH THREE DEGREES OF FREEDOM

This diagram is of the three degrees of freedom longitudinal case. Combining this to a control system, the following diagram results.

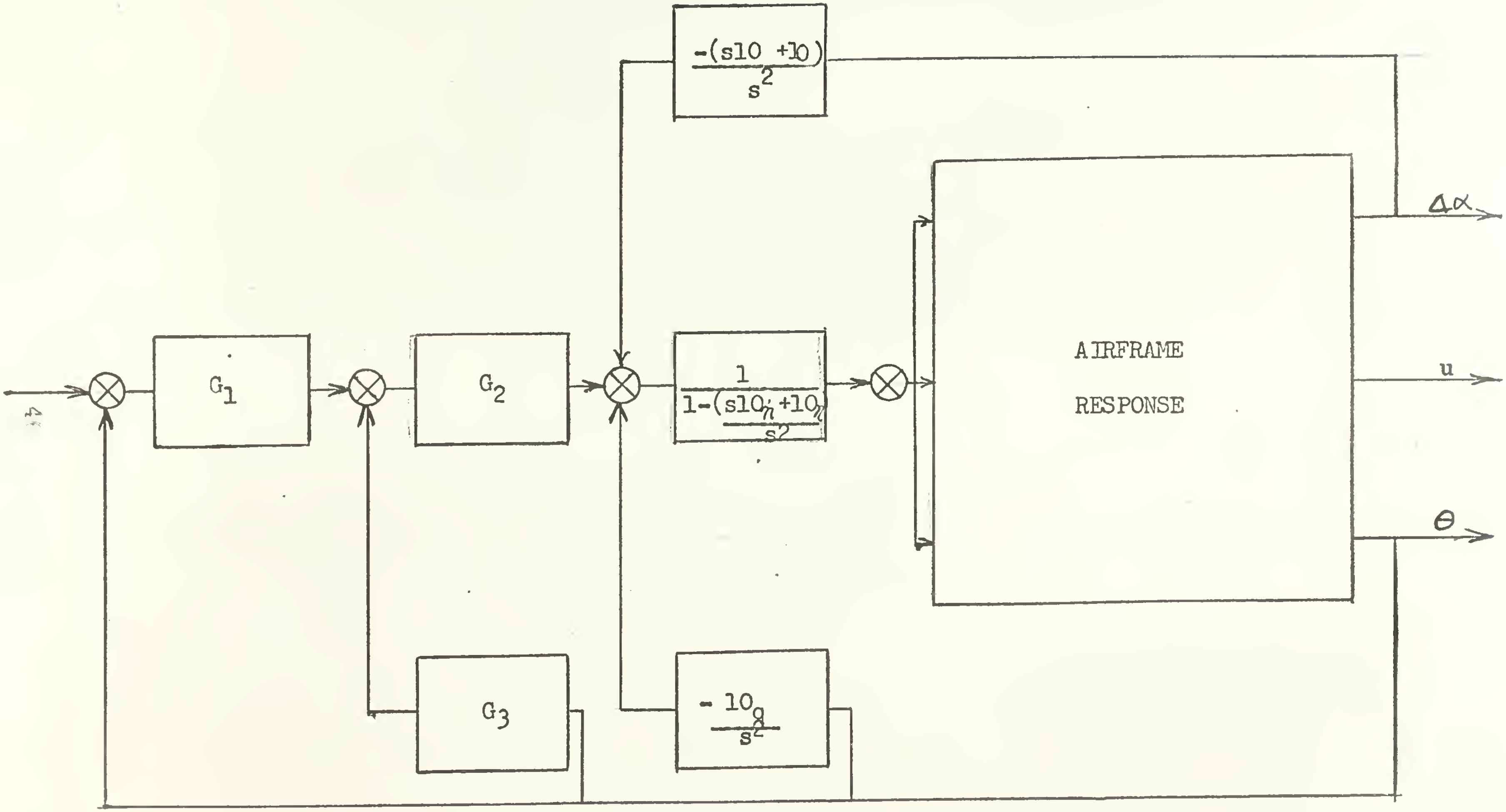


Figure 4-3 COMPLETED LONGITUDINAL AIRFRAME SYSTEM RESPONSE

Thus, the control problem for the simple attitude θ is presented in Fig. 4-2 and 3. The two represent the different point of view of the aeronautical and the electrical engineer although the system is the same. In Fig. 4-2 the aeronautical engineer is faced with the primary problem of compensation within the airframe. For the electrical engineer who inherits the fixed airframe, the attention would shift to Fig. 4-3. In this figure, G_1 would represent the pilot response in moving the control lever. G_2 would represent the servo system response of a hydraulic or electric system. G_3 would represent the error detection and correction signal gained from an electronic element of some kind. Similar situations of block diagramming would exist for the changes in speed and altitude. For the most part, the compensation schemes to achieve stability or improve stability are fixed with respect to paths. The change in aerodynamic coefficients will vary the paths in the airframe response group. If the electronic devices are to be used, then the pick offs must be at the system outputs and the signals fed back to near or at the input or reference signal node point.

In Fig. 4-3 the pilot constitutes the outermost loop. However, this is not mandatory. In a fully automatic system, the pilot circuit is opened. Few paths exist where both aerodynamic or electronic means could be used to achieve stable control. One can be noted on Fig. 4-2. That is elevator position response η which could be altered aerodynamically by changing parts of the present feedback block or adding another electronic control block. It is then fed into the force

input node marked Δ Fe. There is no guarantee that this path solves any stability problem albeit the flutter problem may be aided. It merely serves as an example of two possible types of compensation over the same path.

The lateral systems are not shown or discussed in this chapter but the same principles of control systems apply. It should be remembered that the airframe response shown here covers the small perturbation response concept and thus is greatly simplified. The following sections will deal with expansion of the concept for some workable systems.

5. Expansion of the Standard Form for Greater Disturbances

The system developed by Chu was for linear functions. It is the purpose of this section to explore the equations of motion and determine whether the equations of motion can be used for larger disturbances than considered in Section III. If so, then to what degree can this system be used.

For all practical purposes, the small angle approximation holds quite well up to one-tenth radian. Therefore, $\Theta \simeq \sin\Theta$ and $\cos\Theta \simeq 1$. With this simplification in mind, Equation II-19 is then rewritten.

$$
\begin{aligned}
& m(\dot{U} + QW - RV) + mg\sin\Theta_o - \Psi mg(\cos\Theta_o \sin\phi_o) + \\
& \Theta mg(\cos\Theta_o \cos\phi_o) = X_o + X'_u U + X'_w W + X'_{\dot{w}}\dot{W} + X'_q Q + X'_\eta \eta \\
& + X'_{\dot{\eta}}\dot{\eta} + X'_v V + T_o \cos\Theta_T + T_u \cos\Theta_T U + T_{\delta\,RPM}\,\delta_{RPM} \cos\Theta_T
\end{aligned}
\qquad \text{V-1}
$$

The thrust forces are combined into the forward velocity motion u and a steady state reference condition of

$$
m(Q_o W_o - R_o V_o) + mg\sin\Theta_o = X_o + T_o \cos\Theta_T \qquad \text{V-2}
$$

is assumed.

Subtracting V-2 from V-1, the following equation results

$$
\begin{aligned}
& m(\dot{u} + Q_o w + W_o q + qw - R_o v - V_o r - vr) \\
& - \Psi' mg(\cos\Theta_o \sin\phi_o) + \Theta mg(\cos\Theta_o \cos\phi_o) - X'_u \\
& - X_w - X'_{\dot{w}}\dot{W} - X'_q Q - X'_\eta \eta - X'_{\dot{\eta}}\dot{\eta} - X'_v V = 0
\end{aligned}
\qquad \text{V-3}
$$

The product term qw is dropped as a small value product situation. The same applies to the vr term.

Converting to La Place notation and dividing by the mass the following equation will result

$$
\begin{aligned}
&su + \dot{\Theta}_o U_o \Delta\alpha + sW_o \Theta - \dot{\Psi}_o U_o \beta - \Psi g \cos\Theta_o \sin\phi_o \\
&+ \Theta g \cos\Theta_o \cos\phi_o - X_u u - (sX_{\dot{w}} + X_w)\Delta\alpha - sX_q \Theta \\
&- (sX_{\dot{\eta}} + X_\eta)\eta - X_\beta \beta = 0
\end{aligned}
\qquad \text{V-4}
$$

The y and z direction force equations will form up in the same manner.

$$
\begin{aligned}
&sU_o \beta + \dot{\Psi}_o u + U_o \dot{\Psi} - \dot{\phi}_o U_o \Delta\alpha - W_o \dot{\phi} - \Psi g \sin\Theta_o \\
&- g(\cos\Theta_o \sin\phi_o) - g\phi \cos\Theta_o \cos\phi_o + Y_\beta \beta \\
&- sY_p \phi - sY_r \psi - (Y\xi)\xi - (sY\dot{\zeta} + Y\zeta)\zeta = 0
\end{aligned}
\qquad \text{V-5}
$$

$$
\begin{aligned}
&sU_o \Delta\alpha + P_o U_o \beta + V_{os}\phi - Q_{ou} - V_{os}\Theta + g \sin\Theta_o \Theta \\
&+ g \cos\Theta_o \sin\phi_o \phi - g \cos\Theta_o \cos\phi_o - Z_u U \\
&- (sZ_{\dot{w}} + Z_w) U_o \Delta\alpha - sZ_q \Theta - (sZ_{\dot{\eta}} + Z_\eta)\eta = 0
\end{aligned}
\qquad \text{V-6}
$$

The moment equations can be formed in much the same manner. From Section Two, Equation II-22 would convert to this equation:

$$s^2\phi + \left(\frac{C-B}{A}\right)(sQ_o\psi + sR_o\theta) + \frac{E}{A}(sP_o\theta - sQ_o\psi) - \frac{E}{A}s^2\psi$$

$$= (sL_{\dot\beta} + L_\beta)\beta - L_{ps}\phi - (sL_r\psi) - (sL_{\dot\xi} + L_\xi)\xi - L_\zeta\zeta \qquad \text{V-7}$$

In a similar manner Equations II-23 and II-24 can be brought to a similar maximum linear condition. They are shown in the following equations:

$$s^2\theta + \left(\frac{A-C}{B}\right)(sP_o\theta + sQ_o\phi) + \frac{2E}{B}(sP_o\phi - sR_o\psi)$$

$$= M_u u - (sM_{\dot w} + M_w)\, U_o\Delta\alpha - s(sM_{\dot q} + M_q)\theta - (sM_{\dot\eta} + M_\eta)\eta \qquad \text{V-8}$$

$$s^2\psi + \left(\frac{B-A}{C}\right)(sP_o\theta + sQ_o\phi) + \frac{E}{C}(sQ_o\psi + sR_o\theta - s^2\phi)$$

$$= N_\beta\beta - sN_p\phi - sN_R\psi - N_\xi\xi - (sN_{\dot\zeta} + N_\zeta)\zeta \qquad \text{V-9}$$

Also the control hinge moment equations III-9, III-31 and III-32 are recalled in their same form.

These nine equations represent the linear equations of motion of the airframe and include all cross-coupling effects. These equations still retain the requirement of a linear approximation. The equations with the author's modification of adding the equations $-1 + 1 = 0$ make up Table V-I. This is an 18 x 18 determinant. From this table the block diagram of Fig. 5-1 can be drawn.

	η		$\Delta\alpha$		θ		u		ζ		ψ		β		ϕ		ξ
$1-\frac{(s1_{0\dot{\eta}}+1_{0\eta})}{s^2}$	0	$\frac{-(s1_{0\dot{\alpha}}+1_{0\alpha})}{s^2}$	0	$\frac{-s1_{0q}}{s^2}$	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{-(sZ_{\dot{\eta}}+Z_{\eta})}{s}$	$1-\frac{(sZ_{\dot{w}}+Z_w)}{s}$	0	$\frac{-\left[\begin{matrix}sU_o-g\sin\theta_o\\+sZ_q\end{matrix}\right]}{sU_o}$	0	$\frac{-(Q_o+Z_u)}{sU_o}$	0	0	0	0	0	$\frac{P_o}{s}$	0	$\frac{s\frac{V_o}{U_o}+\frac{g\cos\theta_o\sin\phi_o}{U_o}}{s}$	0	0	0
0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{-(sM_{\dot{\eta}}+M_{\eta})}{s^2}$	0	$\frac{-(sM_{\dot{w}}+M_w)U_o}{s}$	$1+\frac{s(\frac{A-C}{B}P_o-M_q)}{s^2}$	0	$\frac{-M_u}{s^2}$	0	0	0	$\frac{-s\frac{2E}{B}R_o}{s^2}$	0	$\frac{-M_{\beta}}{s^2}$	0	$\frac{s(\frac{A-C}{B}Q_o+\frac{2E}{B}P_o)}{s^2}$	0	0	0
0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	$\frac{-(sX_{\dot{\eta}}+X_{\eta})}{s}$	0	$\frac{U_o(\dot{\theta}_oU_o+(sX_{\dot{w}}+X_w))}{s}$	0	$\frac{sW_o+g\cos\theta_o\cos\phi_o-sX_q}{s}$	$1-\frac{X_u}{s}$	0	0	0	$\frac{-g\cos\theta_o\sin\phi_o}{s}$	0	$\frac{-(\dot{\psi}U_o+X_{\beta})}{s}$	0	0	0	0	0
0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	$1-\frac{(s1_{2\dot{\zeta}}+1_{2\zeta})}{s^2}$	0	$\frac{-s1_{2r}}{s^2}$	0	$\frac{-1_{2\beta}}{s^2}$	0	$\frac{-s1_{2p}}{s^2}$	0	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	$\frac{s(\frac{B-A}{C}P_o+\frac{E}{C}R_o)}{s^2}$	0	0	0	$\frac{-(sN_{\dot{\zeta}}+N_{\zeta})}{s^2}$	$1+\frac{s(Q_o\frac{E}{C}-N_r)}{s^2}$	0	$\frac{-N_{\beta}}{s^2}$	0	$\frac{s(\frac{B-A}{C}Q_o-s\frac{E}{C}-N_p)}{s^2}$	0	$\frac{-N_{\xi}}{s^2}$	0
0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
0	0	0	$\frac{-\dot{\phi}_o}{s^2}$	0	0	0	$\frac{\dot{\psi}_o}{U_os^2}$	0	$\frac{-(sY_{\dot{\zeta}}+Y_{\zeta})}{U_os^2}$	0	$\frac{s(U_o-Y_r)-g\sin\theta_o}{U_os^2}$	$1-\frac{Y_{\beta}}{s^2}$	0	$\frac{s(Y_p-W_o)+g\cos\theta_o\cos\phi_o}{s^2U_o}$	0	$\frac{-Y_{\xi}}{U_os^2}$	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
0	0	0	0	0	$\frac{-s(\frac{C-B}{A}R_o-\frac{E}{A}P_o)}{s^2}$	0	0	0	$\frac{-L_{\zeta}}{s^2}$	0	$\frac{-s\left[\begin{matrix}(\frac{C-B}{A}+\frac{E}{A})Q_o\\-s\frac{E}{A}-L_r\end{matrix}\right]}{s^2}$	0	$\frac{-(sL_{\dot{\beta}}+L_{\beta})}{s^2}$	$1-\frac{sL_p}{s^2}$	0	$\frac{-(sL_{\dot{\xi}}+L_{\xi})}{s^2}$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	$\frac{-s1_{1r}}{s^2}$	0	0	0	$\frac{-s1_{1p}}{s^2}$	$1-\frac{(s1_{1\dot{\xi}}+1_{1\xi})}{s^2}$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1

TABLE 5-I DETERMINANT ARRAY OF AIRFRAME WITH LINEAR APPROXIMATIONS FOR SIX DEGREES OF FREEDOM

FIGURE 5-1 BLOCK DIAGRAM OF THE AIRFRAME AND CONTROL SYSTEM USING THE STANDARD FORM

X
XI
ψ XII
XII
XIV
XV
XVI
ϕ
ϕ R
G3(s)
Δ Fa
XVII
XVIII ξ
1
-s 12 r / s2
Po / s
-sLp / s2
-sl2p / s2
s 11p / s2
+ s 11 r / s2
1 / s2
-Yξ / Uos2
-Nξ / s2
Delete

This block diagram then represents the complete airframe response maintaining linear approximations. Even then this system has limitations. As an example, consider some airplane flying with a steady state wing angle of attack of 15 degrees. (assume a "convention" wing and the reference axis is the zero lift line) Any pilot will quickly testify that an increase of 5 degrees angle of attack at that point will probably result in a rather wildly oscillating maneuver popularly known as a spin. This condition is generally outside of the range of the aerodynamic forces and moments usually computed. Some aerodynamic coefficients such as lift (C_L) are decidedly non-linear in this region. Therefore, the restrictions imposed on this derivation of terms must include a restriction that the airframe is not operating at the very edge of its flight envelope.

The system has been diagrammed directly from Table V-1. The terms in Column One are all connected at node I of Fig. 5-1. Likewise, the terms in Column Two of Table V-1 are all connected to node II. Extra nodes appear in the beginning and end of the block diagram. These are inserted as provisions for control functions. These control nodes are lettered alphabetically. The block diagram contains the required loops for aileron and elevator control. The rudder control loop can be easily inserted when desired.

Examination of Fig. 5-1 reveals many interesting points heretofore known but sometimes only by intuitive reasoning. The concept of separate systems for the symmetric axis and asymmetric axis

systems is very well shown. The level flight steady state conditions used in Section Three stated that

$$Po = Qo = Ro = Vo = Wo = 0$$

Examining the paths between nodes VIII and IX reveal that all of the performance functions are zero under these specified conditions of steady state flight. There is no direct, feedforward or feedback paths between nodes VIII and IX. Thus the restricted case of steady state level flight conditions with small perturbation is clearly divided into the two systems. The longitudinal system includes the forward velocity, vertical velocity, pitching motion and elevator position. The block diagram for the system can be further restricted to two degrees of freedom by holding the X velocity change, u, at zero. This allows change in vertical motion and pitch. This would then be Figs. 3-6 and 3-7 of Section III.

The so-called lateral airframe response system comprises the nodes from IX to XVIII. The rudder and aileron motion, sideslip, yaw and roll comprise the motion for this sytem. With the steady state level flight conditions previously mentioned, the block diagram is identical to Fig. 3-8 of Section III.

An interesting note of these separate systems is the cross coupling terms that begin to arise with larger disturbances. An important one is the pitch due to sideslip. The effects are labeled M_{β} in the diagram. Several aircraft are subject to this. Usually, this is a tucking or nose down motion that appears as sideslip angle (β)

increases. In the simplified systems this effect was neglected. It is of negligible value for very small disturbances, but can become noticeable by the time the sideslip angle (β) reaches five degrees. The effect exists and is shown by this expanded system.

Excellent methods of analysis are available for these standard forms brought forth in this work. Methods of analyzing outputs in response to one or more inputs are available. Work by Thaler[10] contains a very useable system.

Where $N \triangleq$ the output signal at the n th node

$\Delta \triangleq$ characteristic determinant

$\Delta_{a_n} \triangleq$ the cofactor of the determinant with "a" the node location receiving the input and "n" the node location of the output signal.

$G_{1a} \triangleq$ the performance function on the input signal prior to entering the input node.

$I \triangleq$ the input signal

$G_{nN} \triangleq$ the direct path performance function between the output node and the output signal.

Thus a single input signal would yield an output signal

$$N_I = \frac{G_{nN}}{\Delta}\left[\Delta_{a_n} G_{1a} I\right] \qquad \text{V-11}$$

For two or more input signals additional subscripts would be used and the functions are additive. The rules of superposition of linear outputs justify this method.

$$N = \frac{G_{nN}}{\Delta}\left[\Delta_{a_n} G_{1a} I + \Delta_{b_n} G_{2b} II + \Delta_{c_n} G_{3c} III + ----\right] \qquad \text{V-12}$$

This equation would give the resulting output of one or more inputs at various nodes in the system.

The degrees of freedom can be restricted also. In the block diagram, (Fig. 5-2) consider a cut made in the path at point "s".

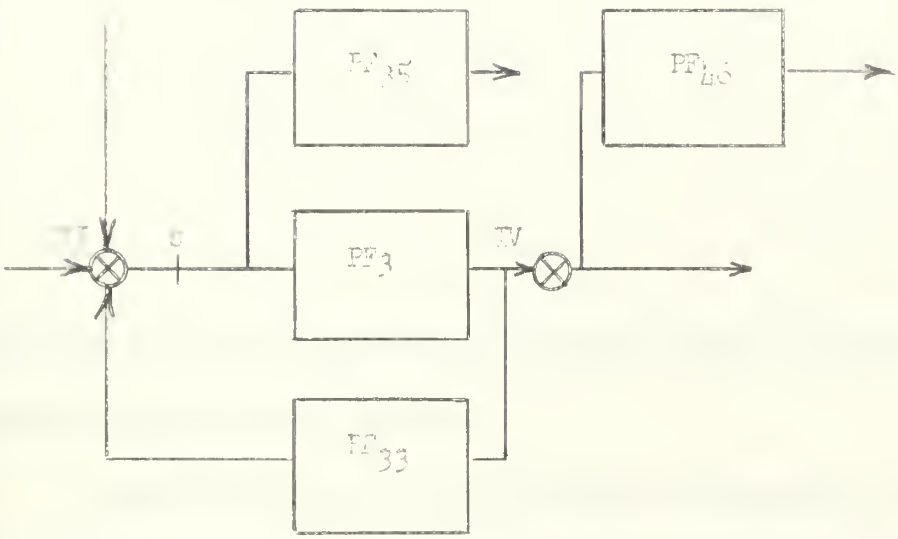


Figure 5-2 ILLUSTRATION OF RESTRICTING FREEDOM OF MOVEMENT IN A SYSTEM BY CUTTING A NODE OUTPUT IN THE BLOCK DIAGRAM

This opening in the path now eliminates the signal to the direct path, the feedback and the feedforward path. The result of this in the determinant is the insertion of zeros in all the performance functions in the column. It leaves only the constant, one, in the main diagonal block. The reader will recall that the main diagonal has the term $1 + PF_3 PF_{33}$. This is shown below as opening the output path from the third node.

$$\begin{matrix} x & x & 0 & x & x & x \\ x & x & 0 & x & x & x \\ x & x & 1 & x & x & x \\ x & x & 0 & x & x & x \\ x & x & 0 & x & x & x \\ x & x & 0 & x & x & x \end{matrix}$$

The output path from node three is opened at point s and the determinant is changed in this manner.

A special case occurs for the diagram illustrated in Fig. 5-7. The second node has only one input signal. Because the path was cut at point "s", the output from the second node will also be zero. Thus all feedforward values from this node will be zero. The direct and feedback paths of the second node are non-existant to begin with and remain so. The determinant form would appear as now shown.

$$
\begin{matrix}
x & x & 0 & 0 & x & x \\
x & x & 0 & 0 & x & x \\
x & x & 1 & 0 & x & x \\
x & x & 0 & 1 & x & x \\
x & x & 0 & 0 & x & x \\
x & x & 0 & 0 & x & x
\end{matrix}
$$

Simple manipulation now reduces the six by six determinant to a four by four determinant. Therefore, when it is desired to consider a problem with one or more degrees of freedom held at zero, the simple method of opening the output signal path from the node and placing a zero for all the performance functions in the column leaving only the constant value one in the main diagonal point accomplishes this.

Referring to Section II, the reader can observe the general equations of motion for mass particles and examine one such as Equation II-22 which is repeated below.

$$A\dot{P} + a_1\dot{K}_1 + QR\,(C-B) - E\,(PQ + \dot{R}) + cK_3Q - bK_2R =$$

$$L_o + L'_v v + L'_{\dot{v}}\dot{v} + L'_p p + L'_r r + L'_\xi \xi + L'_{\dot{\xi}}\dot{\xi} \qquad \text{V-13}$$

$$+ L'_\zeta \zeta + L'_{\dot{\zeta}}\dot{\zeta} = 0$$

Note the third term QR is a produce of $(Q_o + q)\,(R_o + r)$. The product when multiplied is $Q_oR_o + Q_o r + R_o$ + qr. From the equations of Section I:

$$Q = \dot{\Theta}\cos\phi + \Psi\sin\phi\cos\Theta \qquad \text{V-14}$$

$$r = \dot{\psi} \cos \phi \cos \Theta - \dot{\Theta} \sin \phi \qquad \text{V-15}$$

When the small angle approximation is no longer valid, the handling of this product term becomes rather involved. Furthermore, the aerodynamic coefficients of the L_v', L_p', L_r' etc terms lose their linear approximations over large changes in flight conditions. Most aerodynamic coefficients are of a decidedly non-linear nature over the full span of an aircraft's maneuvering capability. The coefficients are definitely non-linear with Mach number when the velocity range covers the subsonic, transonic and supersonic spectrum. The reader can observe the general equations of motion in Section II. Considering the complexities of incorporating equations V-14 and V-15 for large movements and the non linearities of the aerodynamic coefficients, it is not long before one realizes that the non linearities are numerous and cover all the equations. The prospect of block diagramming for the purpose of placing all non linearities in one block to be handled by a describing function scheme is hopeless.

The possibility of doing the problem with an analog computer and using no approximations is fair. Undoubtedly the task, if attempted, will be of considerable magnitude.

Thus, it is seen that this system of standard determinant arrays and standard block diagrams can be used to handle the airframe problem as long as a linear approximation is maintained. The system is flexible and can carry varied conditions of multiple inputs. It can also be noted

that for the six degrees of freedom problem over a large variation, it is of limited value except for diagramming a computer problem.

6. Prospect and Retrospect

The airframe motion problem has been derived quite generally in the first two sections of this work. The attempt was made to maintain as many effects as possible in the general equations of motion. This was done so that the basic equations of motion could be used in a variety of airframes varying from missiles to ground effect machines. In the succeeding sections the author has confined his discussions to equations of basic airplane types. Rotor effects were disregarded and thrust effects were combined with drag to determine the effects along the X axis system. This should not imply that the equations of motion cannot be used for other airframe types. They can in all cases where the assumptions of a rigid body, constant mass, etc. are still valid.

The author has several thoughts on areas of further exploration. He will discuss them briefly and mention any work of possible interest in connection with this field.

The primary aim for control engineering in this field is the development of an orderly process for multiloop compensation. It is shown clearly in Fig. 5-1 that several paths of possible compensation exist. What path or paths are the most successful and/or easiest to achieve multiloop compensation? Some work has been done in this field. Specifically, an electronic compensation of a missile using this determinant method was done by Anderson and Roane.[11] They confined their efforts to the roll and yaw coupling problem for a specific vehicle. Their efforts achieved a satisfactory solution to meeting the missile

specifications, but no definite method of analysis as to the best or poorest path to compensate a multiloop system was generated. A possible approach would be an examination of aerodynamic coefficients and their known effects. In chapter ten of Perkins and Hage[5] there is a discussion of the fact that drag or coefficient of drag markedly affects the damping in the longitudinal phugoid motion of an airplane. The airframe short period is affected largely by the elevator hinge moment coefficients $C_{h_{\Delta\alpha}}$ and $C_{h_{\eta}}$. An examination of these factors and others might lead to a pattern which could successfully solve the problem for any multiloop system.

The block diagram development should provide a basis for analog simulation of the linear problem expressed in sections III and V. The analog computer arrangement may be further expanded to include some of the non-linear effects using function generators. This problem, allowing six degrees of freedom, would overtax the capacity of the analog equipment presently available at this institution. If an analog computer with a 100 amplifier capacity and 50 function generators was provided, the problem could be nicely handled. Most assuredly, combinations can be made by grouping various functions which will cut the above numbers by one half. However, various outputs will not be available for analysis and many signals which are not physically realizable in the actual system will appear as outputs.

The high speed digital computers could be utilized in a project of statistical evaluation of compensation schemes. The computer's

capability of producing hundreds of solutions per minute would allow examination of data to determine trends and patterns for various compensator schemes. A successful compensation criteria might be derived in this manner.

An aerodynamics group may examine four items of interest in this field. The control surface hinge moment equations used in this work were extremely simplified. Exploration of these effects to attain more refinement would be justified.

In section II, the author pointed out the absence of computational methods for various stability derivatives. Further, the assumption of a quasi-steady state condition is made in computing the known derivatives. A study by Etkin[12] cites an improved system to attain better accuracy of the aerodynamic stability derivates. Further work in this area appears justified.

There is also a need to determine the effects of aeroelasticity on the problem. To this end, instrumentation and flight testing an airframe might prove useful. The capability of the Naval Air Facility to maintain and operate many types of military aircraft should permit use of some airplane or aircraft for this work. An additional use would checking the effects of automatic stabilization equipment in use on the airframe. The HSS-1 helicopter has an excellent control system for this type of research. However, the equations of motion used in this work will require some refinement to control the varying inertia problem caused by the tilting rotor disk.

A final topic for consideration is that of signal flow theory. This idea, which has been advanced in recent years, may be compatible to the same "standardized" system as presented here for the block diagram method.

7. Conclusion

It has been conclusively shown that Chu's "standard" block diagram and determinant method can be applied to a linear system other than a servo control system. With some modification, any set of linear differential equations can be block diagrammed into a standard form for general analysis. A very workable system exists for analysis of a multi-loop control system using one or more input signals. The airframe stability problem can be examined for the linear approximation using this very effective system.

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APPENDIX I

BASIC EQUATIONS OF MOTION

The basic concepts of mass in motion and the equations of motion can be derived in vector notation. This is done using mass particles in motion and determining their effects in two separate axis systems. The basic reference system shown in Fig. AI-I is the starred axis. This is the non-rotating reference system and is also known as the Galilean Axis. The axis system is fixed in the rigid mass but may rotate about the fixed space axis system. It is the relative system. A special set of notation is used here in this appendix which differs from the body of this work.

The definition of the special terms in this appendix is as follows.

$\vec{i}$ $\triangleq$ the vectorial unit along the X axis

$\vec{j}$ $\triangleq$ the vectorial unit along the Y axis

$\vec{k}$ $\triangleq$ the vectorial unit along the Z axis

$\vec{i}^{\,*}$ $\triangleq$ the vectorial unit along the X^* axis

$\vec{j}^{\,*}$ $\triangleq$ the vectorial unit along the Y^* axis

$\vec{k}^{\,*}$ $\triangleq$ the vectorial unit along the Z^* axis

0 $\triangleq$ the origin of the relative axis system

0^* $\triangleq$ the origin of the space axis system

$\vec{R}$ $\triangleq$ the vectorial distance from the space axis origin to the relative axis origin

$\vec{r}$ $\triangleq$ the vectorial distance from the space axis origin to a mass particle in space

$\vec{\rho}$ $\triangleq$ the vectorial distance from the relative axis origin to the mass particle

dm $\triangleq$ the mass particle

$\vec{\rho}_c$ $\triangleq$ the vectorial distance from the mass center to the origin of the relative axis system

$\vec{\Omega}$ $\triangleq$ rotational velocity vector of the relative axis system about the space axis; an absolute velocity vector

$\vec{\omega}$ $\triangleq$ absolute rotational velocity vector of a rotor whose axis is attached to XYZ axis system

$\vec{L}$ $\triangleq$ momentum vector for the entire mass body

$\vec{H}$ $\triangleq$ moment of momentum vector with respect to the space axis

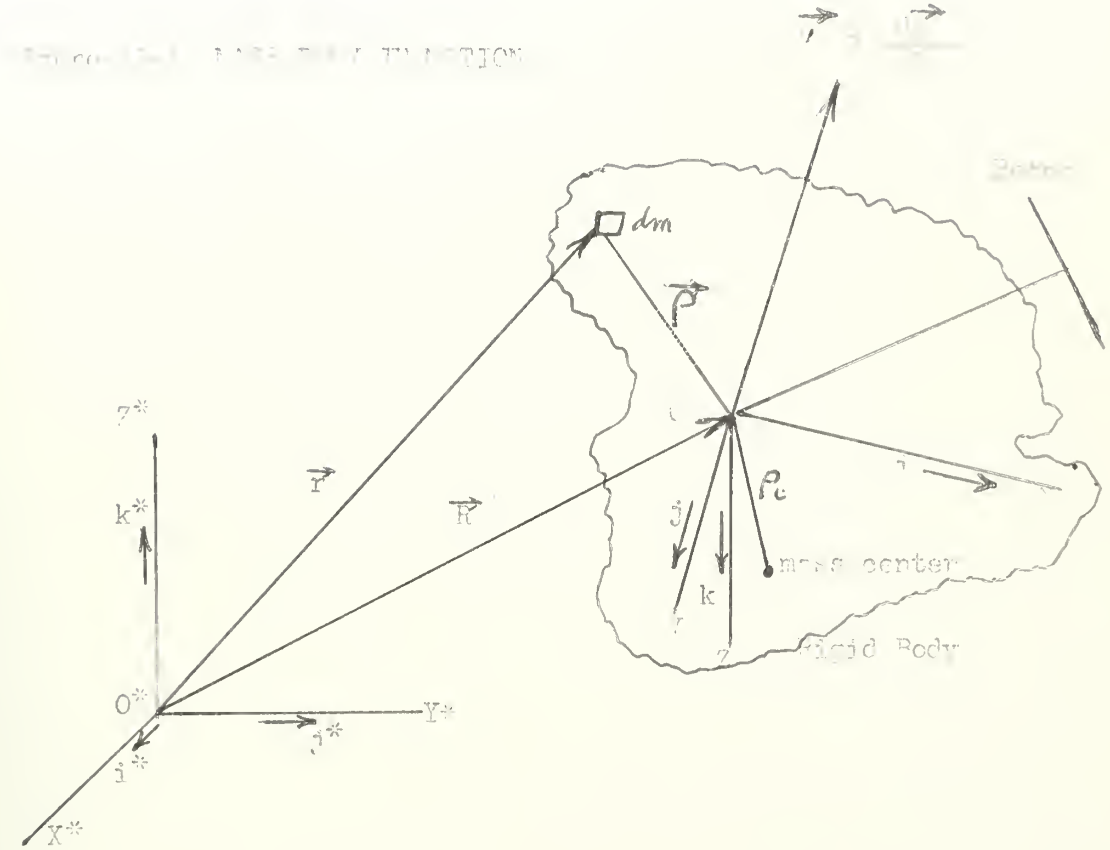
$\vec{G}$ $\triangleq$ moment vector with respect to the relative axis

$\vec{F}$ $\triangleq$ the summation of the external forces on a body

f $\triangleq$ the summation of the internal forces on a body

This appendix will derive the basic equations of motion for any mass moving in space. After the basic equations have been developed, the effects of control actions will be added. The final equations will then be converted to standard aeronautical terminology which is used in the body of the work for aircraft stability considerations.

These equations have no doubt been derived several times in the past one hundred years. However, the author was unable to find any clear derivations in any of the reference material covered. Therefore, the derivations in this Appendix are submitted for the reader's interest and for background information.



Consider

$$\vec{\rho} = \vec{i}x + \vec{j}y + \vec{k}z$$

$$\vec{\omega} = \vec{\Omega} = \vec{i}\omega_1 + \vec{j}\omega_2 + \vec{k}\omega_3$$

Momentum vector for the whole body $\vec{L} = \int dm \frac{dr}{dt}$

$$\frac{\vec{dr}}{dt} = (\vec{V_o} + (\vec{\omega} \times \vec{\rho}) + \dot{\vec{\rho}} \text{ of rotor})$$

$$\vec{L} = \int dm \frac{\vec{dr}}{dt} = \int dm (\vec{V_o} + (\vec{\omega} \times \vec{\rho}) + \dot{\rho}_{\text{rotor}})$$

$$\frac{d\vec{L}}{dt} = \int dm \frac{d^2\vec{r}}{dt^2} = \int \vec{F}$$ but summation of internal forces equals zero.

Hence

$$\frac{d\vec{L}}{dt} = \int dm \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

For an airframe with a large rotor (ie helicopter, STOL aircraft or GEM) consider addition of the rotor to the airframe using the mass center of the aircraft as the origin of the axis system.

$$\frac{d^2r}{dt^2} = \underbrace{\frac{d^2R}{dt^2} + \dot{\vec{\Omega}} \times \vec{\rho} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\rho})}_{\text{transport terms}} + \underbrace{\ddot{\vec{\rho}}}_{\substack{\text{relative}\\ \text{acceleration}\\ \text{term}}} + \underbrace{2\vec{\Omega} \times \dot{\vec{\rho}}}_{\substack{\text{coriolis}\\ \text{effect}}}$$

$\vec{V}_c =$ [illegible]

Consider

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{V}_c}{dt} + \dot{\vec{\Omega}} \times \vec{\rho} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\rho}) + \ddot{\vec{\rho}} + 2\vec{\Omega} \times \dot{\vec{\rho}}$$

Then

$$\frac{d^2\vec{r}}{dt^2} = \vec{i}^*\dot{V}x + \vec{j}^*\dot{V}y + \vec{k}^*\dot{V}z + \vec{i}^*(\Omega_2 Vz - \Omega_3 Vy) + \vec{j}^*(\Omega_3 Vx - \Omega_1 V_3)$$

$$+ \vec{k}^*(\Omega_1 Vy - \Omega_2 Vx) + 0 + 0 + 0 + 0 \text{ where } \rho = 0$$

Hence $F = m \dfrac{d^2\vec{r}}{dt^2}$

$$\left.\begin{aligned} \vec{i}^*Fx &= \vec{i}^*(\dot{V}x + \Omega_2 Vz - \Omega_3 Vy)m \\ \vec{j}^*Fy &= \vec{j}^*(\dot{V}y + \Omega_3 Vx - \Omega_1 V_3)m \\ \vec{k}^*Fz &= \vec{k}^*(\dot{V}z + \Omega_1 Vy - \Omega_2 Vx)m \end{aligned}\right\} \quad \text{neglecting rotor effect.}$$

Therefore in scalar form

$$\left.\begin{aligned} Fx &= (\dot{V}x + \Omega_2 Vz - \Omega_3 Vy)m \\ Fy &= (\dot{V}y + \Omega_3 Vx - \Omega_1 V_3)m \\ Fz &= (\dot{V}z + \Omega_1 Vy - \Omega_2 Vx)m \end{aligned}\right\}$$

Concurs with Etkin
4.4.3
Also Duncan Page 64

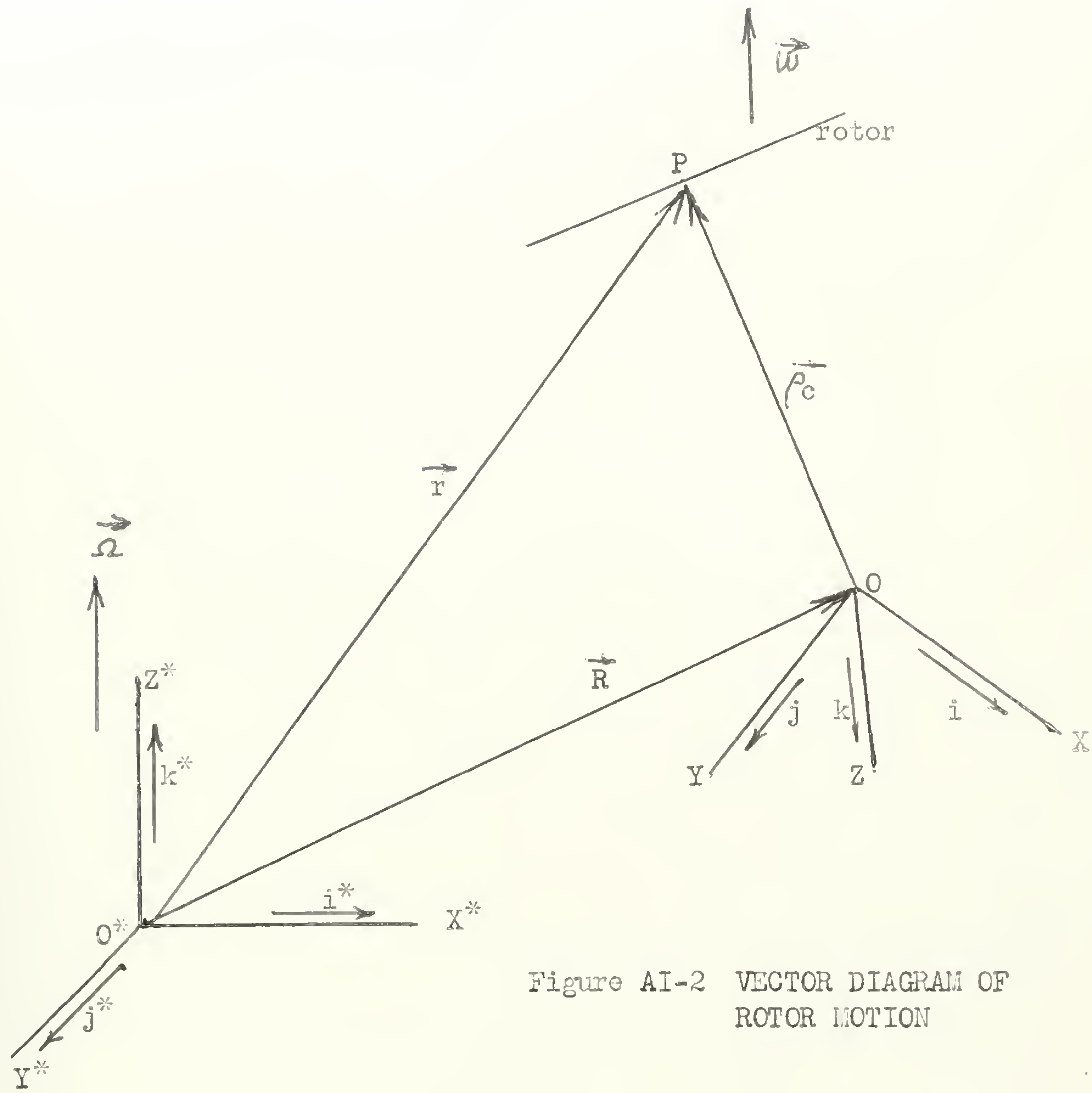


Figure AI-2 VECTOR DIAGRAM OF ROTOR MOTION

ρ_c is fixed in the rigid body. The rotor spins at an absolute angular velocity of but the mass center of the rotor system is at point P. Therefore, $\dot{\vec{\rho}}_c$ and $\ddot{\vec{\rho}}_c = 0$. It is also assumed that $\vec{\Omega}$ = constant. Therefore, the additional term $\Omega \times (\Omega \times \rho_c)$ enters the problem.

$$\vec{\Omega} \times \vec{\rho_c} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ \rho_{c_x} & \rho_{c_y} & \rho_{c_z} \end{vmatrix}$$

$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ is a basic identity, therefore

$\vec{\Omega} \times \vec{\Omega} \times \vec{\rho}_c = (\vec{\Omega} \cdot \vec{\rho_c})\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\vec{\rho_c}$.

Note: coordinates $\vec{i}$, $\vec{j}$, $\vec{k}$ are those in the body and not the inertial or earth system. The values of Ω are primed for the same reason. It is quite possible that ρ_{cy} will be 0 considering alignment of the rotor on the xz plane.

$F_T = F_1 + F_2 = m_1 a + m_2 a \quad F_1 = \text{airframe} \quad F_2 = \text{rotor}$

In resolving this, it must be noted that the mass centers differ from those of the combined masses.

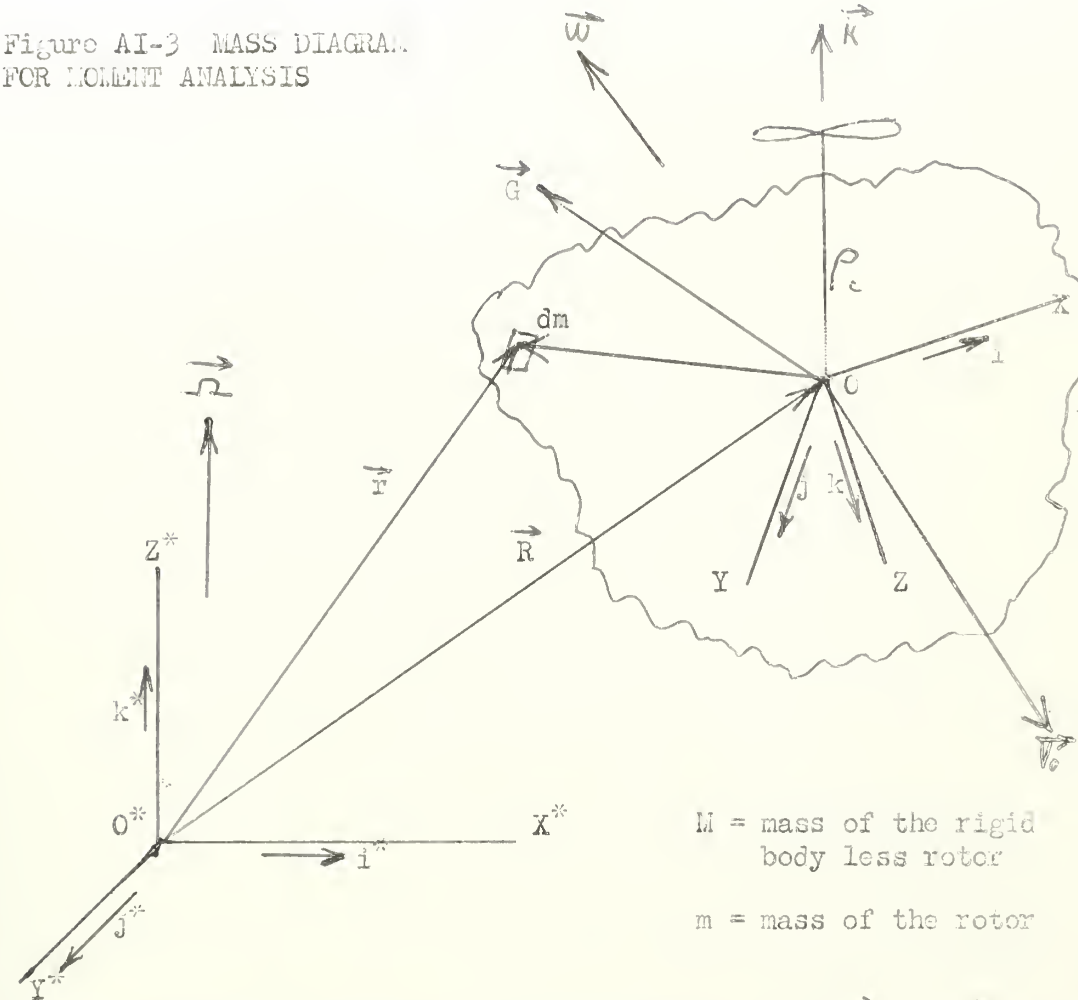
It should be noted that the equations for the acceleration of a particle also contain rotational and relative velocity terms, however, $\ddot{\rho}$ and $\dot{\rho} = 0$ as these particles are in a rigid body. Further, the summation of particles about the mass center makes

$$dm \left[(\dot{\vec{\Omega}} \times \vec{\rho}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{\rho}) \right] \text{ equal to zero.}$$

Moment terms may be considered using Moment of Momentum, $\vec{H}$

From this the differential may be taken and the moment is then derived. Consider $\vec{H}^*$ to be the moment of momentum of the system with respect to the inertial (non-rotating axis). The derivation will include a rotor spinning on the rigid body.

Figure AI-3 MASS DIAGRAM FOR MOMENT ANALYSIS



The absolute angular velocity of the axis OXYZ is $\vec{\Omega} = \vec{i}\,\Omega_1 + \vec{j}\,\Omega_2 + \vec{K}\Omega_3$.

The absolute angular velocity of the body is $\vec{\omega} = \vec{i}\,\omega_1 + \vec{j}\,\omega_2 + \vec{K}\omega_3$.

The relative angular velocity of the body is $|\vec{\omega} - \vec{\Omega}| = n$ (scalar)

The angular motion of a rotor in motion, but attached to the body will be

$$\vec{\omega} - \vec{\Omega} = i(\omega_1 - \Omega_1) + \vec{j}(\omega_2 - \Omega_2) + \vec{K}(\omega_3 - \Omega_3).$$

Henceforth this relative term will be noted as $\vec{K}$ (KAPPA).

The equation for H^* is:

$$H^* = \int \vec{r} \times \frac{d\vec{r}}{dt}\, dm \text{ where } \vec{r} = R + \vec{\rho}$$

Recalling that the momentum equation was $\vec{L} = \int dr\, \frac{d\vec{r}}{dt}$

then

$$\vec{H} = \vec{R} \times \vec{L} + \int \vec{\rho} \times \frac{d\vec{r}}{dt}\, dm$$

The term under the integral will be known as $\vec{H}$.

Recalling that $\frac{d\vec{r}}{dt} = \vec{V}_o + \vec{\Omega} \times \vec{\rho} + \dot{\vec{\rho}}$ (rotor) [illegible]

$$H = \int \left[\vec{\rho} \times (\vec{V}_o + \vec{\Omega} \times \vec{\rho} + \dot{\vec{\rho}}(\text{rotor})) \right] dm$$

This can further be resolved into the moment of momentum about its own Oxyz axis and the velocity moment with respect to the inertial axis.

$$\vec{H} = (M + m)\vec{\rho}_c \times \vec{V}o + \int (\vec{\rho} \times \vec{\Omega} \times \vec{\rho})\, dm + \int \rho \times (K \times \rho)\, dm$$

$$\vec{H}_o \triangleq \int \vec{\rho} \times (\vec{\Omega} \times \vec{\rho})\, dm$$

$$\vec{H}_{o_{rel}} \triangleq \int \vec{\rho} \times (\vec{K} \times \vec{\rho})\, dm$$

To derive moments consider recalling $H^* = \int (\vec{r} \times \frac{d\vec{r}}{dt})\, dm$

$$\frac{d\vec{H}^*}{dt} = \int \left(\frac{dr}{dt} \times \frac{dr}{dt} \right) dm + \int \left[(\vec{R} + \vec{\rho}) \times \frac{d^2\vec{r}}{dt^2} \right] dm$$

$\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0$ and recalling that $\vec{F} = \frac{d\vec{L}}{dt} = \int \frac{d^2\vec{r}}{dt^2}\, dm$

Then substituting these in,

$$\frac{d\vec{H}^*}{dt} = \vec{R} \times \frac{d\vec{L}}{dt} + \vec{C} \quad \text{where } \vec{C} \triangleq \int \left[\vec{\rho} \times \frac{d^2\vec{r}}{dt^2} \right] dm$$

However, we also noted that

$$\vec{H}^* = \vec{R} \times \vec{L} + \int \left(\vec{\rho} \times \frac{d\vec{r}}{dt} \right) dm$$

Therefore:

$$\frac{d\vec{H}^*}{dt} = \frac{d\vec{R}}{dt} \times \vec{L} + \vec{R} \times \frac{d\vec{L}}{dt} + \frac{d\vec{H}}{dt}$$

Equating the two results of $\frac{d\vec{H}^*}{dt}$ it is seen that

$$\cancel{\vec{R} \times \frac{d\vec{L}}{dt}} + \vec{G} = \frac{d\vec{R}}{dt} \times \vec{L} + \cancel{\vec{R} \times \frac{d\vec{L}}{dt}} + \frac{d\vec{H}}{dt}$$

Recall that

$$\frac{d\vec{R}}{dt} = \vec{Vo}$$

and that

$$\vec{L} = (M + m)(\vec{Vo} + \vec{\Omega} \times \vec{\rho}_c + \dot{\vec{\rho}}_c \text{ rotor})$$

$$\text{Now } \frac{d\vec{H}}{dt} = \frac{d}{dt}\left[(M + m)\vec{\rho}_c \times \vec{Vo} + \vec{Ho} + \vec{Ho}_{rel}\right]$$

$$\frac{d\vec{H}}{dt} = (m + M)\left[\frac{d\vec{\rho}_c}{dt} \times \vec{Vo} + \vec{\rho}_c \times \frac{d\vec{Vo}}{dt}\right] + \frac{d}{dt}\left[\vec{Ho} + \vec{Ho}_{rel}\right]$$

combining

$$\vec{G} = \vec{Vo} \times (\vec{Vo} + \vec{\Omega} \times \vec{\rho}_c) M + \vec{Vo} \times (\vec{Vo} + \vec{\Omega} \times \vec{\rho}_c + \dot{\vec{\rho}}_c \text{ rotor}) m$$
$$+ (\dot{\vec{\rho}}_c \times \vec{Vo} + (\Omega \times \vec{\rho}_c) \times \vec{Vo})m + (\dot{\vec{\rho}}_c \times \vec{Vo} + (\Omega \times \rho_c) \times \vec{Vo}) M$$
$$+ (M + m)\vec{\rho}_c \times \frac{d\vec{Vo}}{dt} + \frac{d}{dt}\left[\vec{Ho} + \vec{Ho}_{rel}\right]$$

A further assumption is stated at this point. This is that the point of rotation of the rotor is fixed on the body and remains so. The shaft does not have nutation about some axis but remains fixed on a mount of the rigid body. Therefore, the $\dot{\vec{\rho}}_c$ of the rotating body is zero. Also, logically, in a rigid body the mass does not shift hence $\dot{\vec{\rho}}_c$ is also zero.

In crossing a vector with itself Vo x Vo the result is zero.

$$V_o \times V_o = \begin{vmatrix} i & j & k \\ V_1 & V_2 & V_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \vec{i}\,(V_2V_3 - V_2V_3) + \vec{j}\,(V_1V_3 - V_1V_3) + \vec{k}\,(V_1V_2 - V_1V_3) = 0$$

$$V_o \times V_o = 0$$

Furthermore, crossing two dissimilar vectors in opposite order yields opposite signs and equal magnitude.

$$\vec{V_o} \times \vec{\rho_c} = \begin{vmatrix} i & j & k \\ V_1 & V_2 & V_3 \\ \rho_1 & \rho_2 & \rho_3 \end{vmatrix} = \vec{i}\,(V_2\rho_3 - \rho_2V_3) + \vec{j}\,(\rho_1V_3 - \rho_3V_1) + \vec{k}\,(V_1\rho_2 - \rho_1V_2)$$

$$\vec{\rho_c} \times \vec{V_o} = \vec{i}\,(\rho_2V_3 - \rho_3V_2) + \vec{j}\,(\rho_3V_1 - \rho_1V_3) + \vec{k}\,(\rho_1V_2 - \rho_2V_1)$$

Thus $\vec{V_o} \times \vec{\rho_c} + \vec{\rho_c} \times \vec{V_o} = 0$

Incorporating these two facts, the equation for $\vec{G}$ can be simplified to the following:

$$\vec{G} = \frac{d}{dt}\left[\vec{H_o} + \vec{H_o}_{rel}\right] + (M + m)\left(\vec{\rho_c} \times \frac{d\vec{V_o}}{dt}\right)$$

$\frac{d\vec{H_o}}{dt} = \vec{H_o} + \vec{\Omega} \times H_o$ in a general equation.

Thus it is necessary to solve $\frac{d\vec{H_o}}{dt}$ to find $\vec{G}$.

Considering $\vec{Ho} = \int \left[\vec{\rho} \times (\vec{\omega} \times \vec{\rho})\right]$ which accounts for the absolute velocity.

Recalling an identity of vector algebra:

$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ then

$$\vec{\rho} \times (\vec{\omega} \times \vec{\rho}) = (\vec{\rho} \cdot \vec{\rho})\vec{\omega} - (\vec{\rho} \cdot \vec{\omega})\vec{\rho}$$

$$Ho = \int \Big[(x^2 + y^2 + z^2)(\vec{i}\omega_1 + \vec{j}\omega_2 + \vec{k}\omega_3) - (x\omega_1 + y\omega_2 + z\omega_3)(\vec{i}x + \vec{j}y + \vec{k}z)\Big]\, dm$$

$$= \int \Big[\vec{i}\left[(y^2 + z^2)\omega_1 - xy\,\omega_2 - xz\,\omega_3\right] + \vec{j}\left[(x^2 + z^2)\omega_2 - yz\,\omega_3 - xy\,\omega_1\right] + \vec{k}\left[(x^2 + y^2)\omega_3 - xz\,\omega_1 - yz\,\omega_2\right]\Big]dm$$

The above terms can be recognized in part as moment and product of inertia terms.

$Ix \triangleq \int (y^2 + z^2)dm$

$Iy \triangleq \int (x^2 + z^2)dm$

$Iz \triangleq \int (x^2 + y^2)dm$

$Ixy \triangleq \int (xy)dm$

$Iyz \triangleq \int (yz)dm$

$Ixz \triangleq \int (xz)dm$

Therefore:

$$\vec{Ho} = \vec{i}\,(Ix\,\omega_1 - Ixy\,\omega_2 - Ixz\,\omega_3) + \vec{j}\,(-Ixy\,\omega_1 + Iy\,\omega_2 - Iyz\,\omega_3)$$

$$+ \vec{k}\,(Iz\,\omega_3 - Ixz\,\omega_1 - Iyz\,\omega_2)$$

Considering the effects of the aircraft as a rigid body first

$$\frac{d\vec{Ho}}{dt} = \dot{\vec{Ho}} + \vec{\Omega} \times \vec{Ho} \text{ and for a rigid body } \vec{\omega} = \vec{\Omega}$$

$$\frac{d\vec{Ho}}{dt} = \vec{i}\,(Ix\dot{\Omega}_1 - Ixy\dot{\Omega}_2 - Ixz\dot{\Omega}_3) + \vec{j}\,(-Ixy\dot{\Omega}_1 + Iy\dot{\Omega}_2 - Iyz\dot{\Omega}_3)$$

$$+ \vec{k}\,(-Ixy\dot{\Omega}_1 - Iyz\dot{\Omega}_2 + Iz\dot{\Omega}_3) + \begin{vmatrix} i & j & k \\ \Omega_1 & \Omega_2 & \Omega_3 \\ (Ix - Ixy - Ixz) & (Iy - Ixy - Iyz) & (-Ixz - Iyz + Iz) \end{vmatrix}$$

For any ordinary aircraft or missile, the product of inertia terms Ixy and Iyz can be considered zero because of the mirror symmetry aspect. Ixz may be zero if the principal axis is selected. Thus with no loss in generality, consider Ixy = Ixz = Iyz = 0

$$\frac{d\vec{Ho}}{dt} = \vec{i}\left[Ix\dot{\Omega}_1 + Iz\Omega_2\Omega_3 - Iy\Omega_2\Omega_3\right]$$

$$+ \vec{j}\left[Iy\dot{\Omega}_2 + Ix\Omega_1\Omega_3 - Iz\Omega_1\Omega_3\right]$$

$$+ \vec{k}\left[Iz\dot{\Omega}_3 + Iy\Omega_1\Omega_2 - Ix\Omega_1\Omega_2\right]$$

However, including the effects of Ixz, the product of inertia term,

which is often not zero:

$$\frac{d\vec{H}_o}{dt} = \vec{i}\left[I_x\dot{\Omega}_1 - I_{xz}\dot{\Omega}_3 - I_{xz}\Omega_1\Omega_2 + I_z\Omega_2\Omega_3 - I_y\Omega_2\Omega_3\right]$$
$$+ \vec{j}\left[I_y\dot{\Omega}_2 + I_x\Omega_1\Omega_3 - I_{xz}\Omega_3^2 + I_{xz}\Omega_1^2 - I_z\Omega_1\Omega_3\right]$$
$$+ \vec{k}\left[I_z\dot{\Omega}_3 - I_{xz}\dot{\Omega}_1 + I_y\Omega_1\Omega_2 - I_y\Omega_1\Omega_2 + I_{xz}\Omega_2\Omega_3\right]$$

Using standard aerodynamic terminology, the following items are

defined:

$A \triangleq I_x$

$B \triangleq I_y$

$C \triangleq I_z$

$E \triangleq I_{xz}$

$P \triangleq \Omega_1$

$Q \triangleq \Omega_2$

$R \triangleq \Omega_3$

Thus

$\vec{G} = \frac{d\vec{H}_o}{dt}$ where system is chosen such that $\rho_c = 0$

$$\vec{G} = \vec{i}\left[A\dot{\rho} - E\dot{R} - EPQ + QR\,(C-B)\right]$$
$$+ \vec{j}\left[B\dot{Q} + PR\,(A-C) + E\,(P^2 - R^2)\right]$$
$$+ \vec{k}\left[C\dot{R} - E\dot{P} + PQ\,(B-C) + EQR\right]$$

These equations concur with Duncan, Page 66, and Etkin, Chapter 4

To consider the rotor effects it should be remembered that $\vec{\omega} = \vec{\Omega} + \vec{K}$ where $\vec{K}$ is the relative motion of a rotor on the body. Thus for a rotor there will be a combination of terms considering the absolute angular velocity of the OXYZ system plus the angular velocity of the rotor with respect to the body.

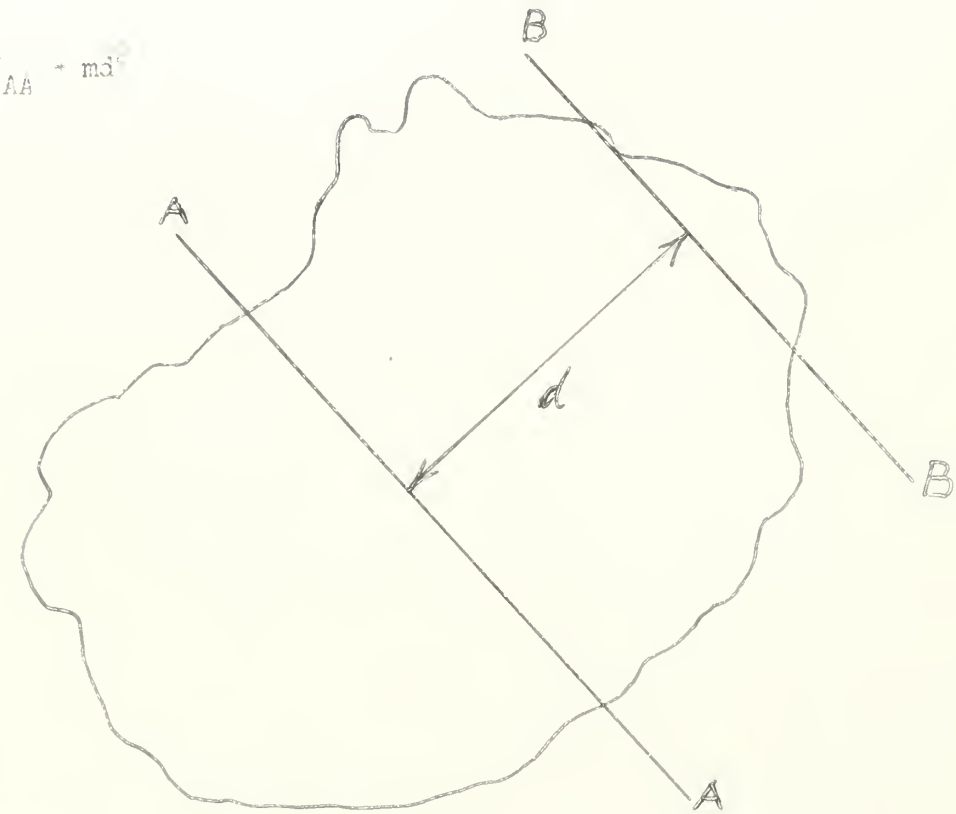
Thus

$$\frac{d\vec{Ho}}{dt} = \dot{\vec{Ho}} + \vec{\omega} \times \vec{Ho} \quad \text{where } \vec{\omega} \triangleq \vec{\Omega} + \vec{K}$$

It should be noted, that for a symmetrical rotor or disk the product of inertia terms are zero where the disk is at least a three bladed propeller. The two bladed propeller has a product of inertia dependent on its angular position. This case will not be considered, but the three or more bladed propeller situation will be used in all considerations. Hence with a dynamically and statically balanced rotor $\dot{\rho}_c = 0$ and the product of inertia is zero.

The moment of inertia of the disk must be considered from the axis of the rigid body. Therefore, the moment of inertia must include the md^2 consideration.

$I_{BB} = I_{AA} + md^2$



The moment of inertia terms for the rotors will be thus defined in the most general sense

$a \triangleq Ix\ \text{rotor} \triangleq Ix + md_1^2$ and aligned with the reference axis

$b \triangleq Iy\ \text{rotor} \triangleq Iy + md_2^2$ and aligned with the reference axis

$c \triangleq Iz\ \text{rotor} \triangleq Iz + md_3^2$ and aligned with the reference axis

$$\vec{G} = \frac{d\vec{H}}{dt} = \frac{d\vec{H}o}{dt} + \frac{d\vec{H}_{rel}}{dt}$$

Recalling the derivation of moments for the rigid body without the rotor, the same method applies.

$$\left.\begin{aligned} \vec{G} &= \vec{i}\,a\dot{\Omega}_1 + \vec{j}\,b\dot{\Omega}_2 + \vec{k}\,c\dot{\Omega}_3 + \vec{\Omega} \times \vec{H}_o \\ \vec{G} &= \vec{i}\,a\dot{\kappa}_1 + \vec{j}\,b\dot{\kappa}_2 + \vec{k}\,c\dot{\kappa}_3 + \vec{\kappa} \times \vec{H}_o \end{aligned}\right\} \quad \text{for a rotor only}$$

Combining this to a system consisting of a rigid body plus a rotor, the following results occur

$$\begin{aligned} \vec{G} = \frac{d\vec{H}}{dt} = \vec{i}\,&\left[I_x\dot{\Omega}_1 - I_{xz}\dot{\Omega}_3 + I_{xz}\Omega_1\Omega_2 + I_z\Omega_2\Omega_3 - I_y\Omega_2\Omega_3\right] \\ + \vec{j}\,&\left[I_y\dot{\Omega}_2 + I_x\Omega_1\Omega_3 - I_{xz}\Omega_3^2 + I_{xz}\Omega_1^2 - I_z\Omega_1\Omega_3\right] \\ + \vec{k}\,&\left[I_z\dot{\Omega}_3 - I_{xz}\dot{\Omega}_1 + I_y\Omega_1\Omega_2 + I_x\Omega_1\Omega_2 + I_{xz}\Omega_2\Omega_3\right] \\ + \vec{i}\,a\dot{\kappa}_1 + \vec{j}\,b\dot{\kappa}_2 + \vec{k}\,c\dot{\kappa}_3 + &\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Omega_1 & \Omega_2 & \Omega_3 \\ a\kappa_1 & b\kappa_2 & c\kappa_3 \end{vmatrix} \end{aligned}$$

Combining and using aerodynamic notation.

$$\begin{aligned} \vec{G} = \vec{i}\,&\left[A\dot{P} - E\dot{R} + a\dot{\kappa}_1 - EPQ + (C-B)QR + c\kappa_3 Q - b\kappa_2 R\right] \\ + \vec{j}\,&\left[B\dot{Q} + b\dot{\kappa}_2 + (A-C)PR + E(P^2 - R^2) + a\kappa_1 R - c\kappa_3 P\right] \\ + \vec{k}\,&\left[C\dot{R} - E\dot{P} + c\dot{\kappa}_3 + (B-A)PQ + EQR + b\kappa_2 P - a\kappa_1 Q\right] \end{aligned}$$

This result concurs with Etkin, page 116.

Therefore, using a standard aerodynamic notation the moment:

$\vec{G} = \vec{i}\,L + \vec{j}\,M + \vec{k}\,N$

and the scalar quantities are as follows

$$[illegible]$$

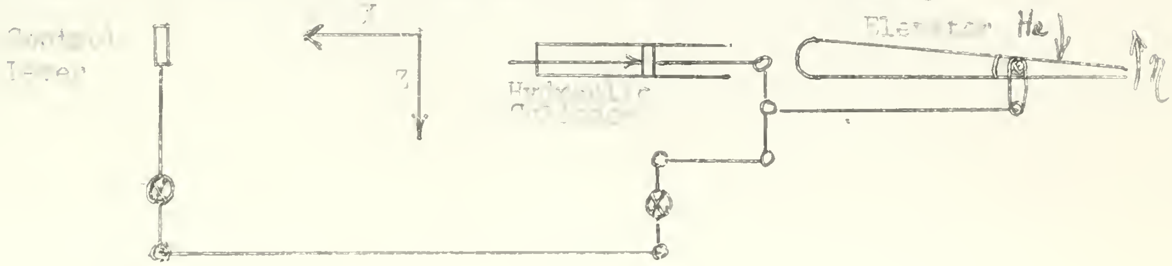
$$[illegible]$$

$$[illegible]$$

Thus, the force and moment equations derived in this appendix are applicable to a rigid body of constant mass where the body is in motion. It also considers the effects of a three bladed or more rotor system which is attached on some fixed point of the rigid body. The rotor effects may be included or discarded depending on the magnitude of rotor inertia and the relative velocity of the rotor.

It should be noted that the derivations are done in a general form, but in the concluding equations, the usual aerodynamic notation is substituted to bring the equations to this specific problem. Furthermore, missile or aircraft shape and density has been considered symmetrical about the xz plane. Thus, the product of inertia terms Ixy and Iyz have been considered zero. No generality is lost by this and if, for some reason, the body is sufficiently unsymmetrical, the terms can be quickly added in.

The following is taken from Chapter 4 of Etkin[4]. The concept of motion on the control surfaces is shown in the following



[illegible] CONTROL SYSTEM

The LaGrange equation of motion in a moving reference frame is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = \mathcal{F}$$

$T \triangleq$ kinetic energy of the system relative to the chosen frame of reference

$\mathcal{F} \triangleq \dfrac{\partial W}{\partial q_K}$ = generalized force

$W \triangleq$ work done on the system by the external forces which act upon it

q_K = generalized coordinate

H = aerodynamic moments

δ = rotation of control surface in radians

Kinetic energy $\quad T = \frac{1}{2} I \dot{\delta}^2 \qquad \dfrac{\partial T}{\partial \dot{q}_K} = I \dot{\delta}$

$$\mathcal{F} = \frac{\partial W}{\partial q_K}$$

For elevator motion

$$\delta W = He\,\delta(\delta e) + Fe\,\delta(\delta e) + \delta Wi$$

$Fe \triangleq$ generalized elevator control force

$\delta Wi \triangleq$ work done by inertia forces

Work done by the control forces is

$$Fe\,\delta(\delta e) = P\,\delta sp + J\,\delta sJ$$

$$Fe = P\,\frac{dsp}{d\delta e} + J\,\frac{dsJ}{d\delta e}$$

$\dfrac{dsp}{d\delta} \triangleq$ gear ratio of pilot's stick to control surface

$\dfrac{dsJ}{d\delta} \triangleq$ gear ratio or advantage of hydraulic cylinder to control surface

Evaluating δW_i

consider $dF_i = \dfrac{d\dot{r}}{dt}\,dm$

recalling from previous derivation that

$$\frac{d\vec{\dot{r}}}{dt} = \frac{d\vec{V}_c}{dt} + \vec{\dot{\Omega}} \times \vec{\rho} + \vec{\Omega} \times \vec{\Omega} \times \vec{\rho} + \vec{\ddot{\rho}} + 2\vec{\Omega} \times \vec{\dot{\rho}}$$

Then

$$dF_{xi} = -\left[\frac{dV_{cx}}{dt} + 2Q\dot{z} - 2R\dot{y} - x(Q^2 + R^2) + y(PQ - \dot{R}) + z(PR - \dot{Q})\right] dm$$

dF_{yi} and dF_z are similar in form.

Looking at an elevator assembly, it can be seen that assuming a lamina in the xy-plane, the displacement in the C_Z direction is amount $\delta(\delta e)$.

The work done by the inertia forces is $\delta W_i = \int \rho_e\,\delta(\delta e)\,dF_{zi}$

$$\frac{\delta W_i}{\delta(\delta e)} = \int \frac{dV_{cz}}{dt}\rho_e\,dm - (PR - \dot{Q})\int \chi\rho_e\,dm - RQ + \dot{P}\int y\rho_e\,dm$$

The coriolis terms $2R\dot{y}$ and $-2Q\dot{x}$ are zero as there is no work done in those directions. Furthermore $z = 0$ because of the assumed lamina.

$$a_{cz} \triangleq \frac{dV_{cz}}{dt} \text{ and } a_{cz}\int \rho_e\,dm = m_e l_e a_{cz}$$

$$(PR - \dot{Q})\int \chi\rho_e\,dm = P_{ex}(PR - \dot{Q})$$

$P_{ex} \triangleq$ product of inertia of the elevator

By symmetry $\int \rho_e \, dm = 0$

$$\frac{\delta T}{\delta(\delta e)} = m_e l_e a_{c_z} - P_{ex}(PR - \dot{Q})$$

$$I_e \ddot{\delta}_e - \mathcal{F} = m_e l_e a_{c_z} - P_{ex}(PR - \dot{Q}) + H_e + F_e$$

$$I_e \ddot{\delta}_e + m_e l_e a_{c_z} + P_{ex}(PR - \dot{Q}) = H_e + F_e \qquad \text{(Etkin 4.8,13)}$$

Similarly for the rudder

$$I_r \ddot{\delta}_r - a_{c_y} m_r l_r - (\dot{R} + PQ) P_{rx} - (RQ - \dot{P}) P_{rz} = H_r + F_a$$

and for the aileron

$$I_a \ddot{\delta} - 2 P_{ay}(RQ + \dot{P}) = 2 H_a + F_a$$

APPENDIX II

ROLL, YAW AND PITCH

The derivation of the yaw, roll and pitch rates is given here to show the actual motion taking place. Frequently the motion is considered small and small angle approximations are used. These will be noted at the conclusion.

First, consider the Eulerian Axis used in the aeronautical field shown below. The rotation takes place in the order of yaw (ψ), pitch (θ), and roll (ϕ).

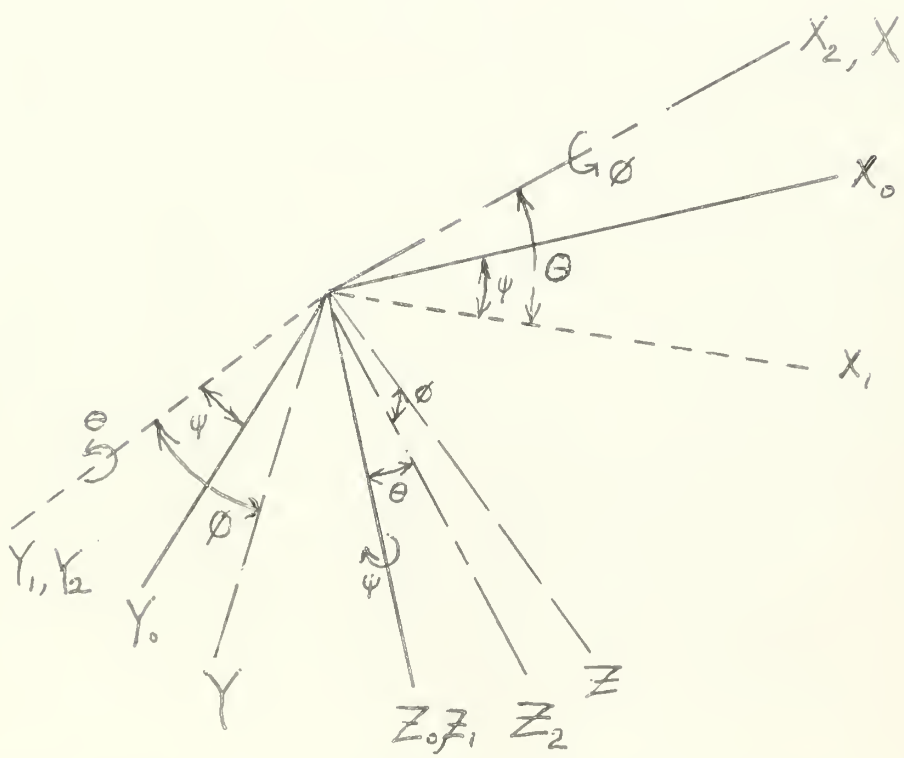


Figure AII-1 EULERIAN AXIS SYSTEM SHOWING ROTATION ABOUT THE Z, Y, AND X AXIS

The initial conditions at X_o, Y_o, and Z_o and the relations of $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$ can be related to P, Q and R of the body axis at x, y, z. In the above diagram consider the angular velocity P which is about the OX axis. It must be the sum of the velocities about the OZ_o, OY_1, and OX_2 axis respectively.

OZ_o makes an angle $\frac{\pi}{2} + \theta$ with OX. The rate thus $\dot{\psi} \cos (\frac{\pi}{2} + \theta) = -\dot{\psi} \sin \theta$. OY_1 is perpendicular with OX so $\dot{\theta} \cos \frac{\pi}{2} = 0$ and OX is parallel with OX_2 or the motion $\dot{\phi} \cos 0 = \dot{\phi}$.

Thus the summation $P = \dot{\phi} - \dot{\psi} \sin \theta$.

To evaluate Q the angle YOZ_o must be determined. Specifically the value $\dot{\psi} \cos \widehat{YOZ_o}$ is desired.

$$\begin{aligned} \cos \widehat{YOZ_o} &= \cos \widehat{YOZ_2} \cos \widehat{Z_2OZ_o} \\ &= \cos (\frac{\pi}{2} - \phi) \cos \theta \\ &= \sin \phi \cos \theta \end{aligned}$$

Thus the contribution of OZ_o axis motion is $\dot{\psi} \sin \phi \cos \theta$. The angle between OY and OY_1 is ϕ so the motion here results as $\dot{\theta} \cos \phi$. OY is perpendicular to OX_2 hence the angular motion $\dot{\phi} \cos \frac{\pi}{2} = 0$.

Thus $Q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta$

In the third angular velocity motion R the yaw motion appears as $\dot{\psi} \cos \phi \cos \theta$ while the motion of pitch is $\dot{\theta} \cos (\frac{\pi}{2} + \phi) = -\dot{\theta} \sin \phi$.

The rolling term vanishes $\dot{\phi} \cos \frac{\pi}{2} = 0$.

Thus $R = -\dot{\theta} \sin \phi + \dot{\psi} \cos \phi \cos \theta$

Hence the relationship of the body yaw, pitch and roll rates (P, Q, and R) to the designated reference system angular velocities $\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$ is:

$$P = \dot{\phi} - \dot{\psi} \sin \theta$$

$$Q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta$$

$$R = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi$$

These results concur with Duncan page 81 and Etkin page 116.

It should be noted that for small angles the approximation below is often used:

$$P \cong \dot{\phi}$$

$$Q \cong \dot{\theta}$$

$$R \cong \dot{\psi}$$

A second consideration which must be made regarding movement about the reference axis is the velocity or displacement with respect to time. It is desired to obtain the velocity with respect to the fixed frame or inertial axis i* x, j*y and k*z.

Consider first that in scalar values

$$\frac{dx^*}{dt} = U_1$$

$$\frac{dy^*}{dt} = V_1$$

$$\frac{dz^*}{dt} = W_1$$

It should be noted, however, that in the three position system shown on the previous page that

$$U_3 = U$$

$$V_3 = V \cos \phi - W \sin \phi$$

$$W_3 = V \sin \phi + W \cos \phi$$

and $$W_2 = -U_3 \sin \Theta + W_3 \cos \Theta$$

$$V_2 = V_3$$

$$U_2 = U_3 \cos \Theta + W_3 \sin \Theta$$

and $$U_1 = U_2 \cos \Psi - V_2 \sin \Psi$$

$$V_1 = U_2 \sin \Psi + V_2 \cos \Psi$$

$$W_1 = W_2$$

Combining the 1, 2 and 3 position terms

$$\frac{dx^*}{dt} = U \cos \Theta \cos \Psi + V (\sin \phi \sin \Theta \cos \Psi - \cos \phi \sin \Psi) + W (\cos \phi \sin \Theta \cos \Psi + \sin \phi \sin \Psi)$$

$$\frac{dy^*}{dt} = U \cos \Theta \sin \Psi + V (\sin \phi \sin \Theta \sin \Psi + \cos \phi \cos \Psi) + W (\cos \phi \sin \Theta \sin \Psi - \sin \phi \cos \Psi)$$

$$\frac{dz^*}{dt} = -U \sin \Theta + V \sin \phi \cos \Theta + W \cos \phi \cos \Theta$$

Thus, the velocity or rate of change with respect to the inertial or earth's reference axis in this case is expressed in these terms as derived. These velocity terms are derived in Etkin Page 102 and coincide with the derivation shown.

Therefore, this appendix accounts for the changes in angular and directional displacement of a rigid body with respect to a fixed reference axis system.

APPENDIX III

NOTES ON DETERMINANT MANIPULATION

The body of this project contains several determinant expansions where the order is doubled. These manipulations do not alter the values of the determinant and the resulting values of the performance functions are the same.

The simple determinant

$$\begin{vmatrix} A1 & B1 & C1 \\ A2 & B2 & C2 \\ A3 & B3 & C3 \end{vmatrix} = \Delta$$

evaluated will yield the following result

$$\Delta = \begin{bmatrix} A1B2C3 + B1C2A3 + C1A2B3 \\ - C1B2A3 - A1C2B3 - B1A2C3 \end{bmatrix} \qquad \text{A3-1}$$

This expansion by the equations $-1 + 1 = 0$ will result in the following determinant.

$$\begin{vmatrix} A1 & 0 & B1 & 0 & C1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & A2 & B2 & 0 & C2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & A3 & 0 & B3 & C3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \Delta$$

The resulting evaluation will yield the same value as result (A3-1). Furthermore, if the terms below the main diagonal are allowed to remain in their original columns in lieu of shifting one place to the

right as is done in the main body, the resulting Δ is still the same.

The results of evaluating cofactors for particular performance functions remain unchanged. For example it is desired to evaluate the cofactor Δ_{12} of the original determinant. Thus:

$$\Delta_{12} = \begin{vmatrix} A1 & B1 & C1 \\ A2 & B2 & C2 \\ A3 & B3 & C3 \end{vmatrix} = (-1)\ [A2C3 - A3C2] \qquad A3-2$$

In the expanded determinant the equivalent cofactor is Δ_{13}

$$\begin{vmatrix} A1 & 0 & B1 & 0 & C1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & A2 & B2 & 0 & C2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & A3 & B3 & 0 & C3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \Delta_{13}$$

$$= \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & A2 & 0 & C2 \\ 0 & 0 & 1 & 0 \\ 0 & A3 & 0 & C3 \end{vmatrix} = - \begin{vmatrix} A2 & 0 & C2 \\ 0 & 1 & 0 \\ A3 & 0 & C3 \end{vmatrix}$$

$$\Delta_{13} = (-1)\ [A2C3 - A3C2] \qquad A3-3$$

The result of Equation A3-3 equals Equation A3-2. With the values below the main diagonal left in their original columns, the following occurs:

$$\Delta_{13} = \begin{vmatrix} A1 & 0 & B1 & 0 & C1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ A2 & 0 & B2 & 0 & C2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ A3 & 0 & B3 & 0 & C3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

By inspection of columns two and six, two reductions are made to the following:

$$\Delta_{13} = (-1) \begin{vmatrix} A2 & 0 & C2 \\ 0 & 1 & 0 \\ A3 & 0 & C3 \end{vmatrix}$$

$$\Delta_{13} = (-1) \left[A2C3 - A3C2 \right] \qquad \text{A3-4}$$

The resulting equation (A3-4) is the same as equation (A3-2).

A similar examination using a four by four determinant system was made. This was done to insure that the simplified effects of a three by three or lower order system did not cover up an error in a four by four or higher order determinant. The same manipulations can be performed and the resultant value of the determinant is not altered.

Thus, the individual has the option of placing his values in either column. The location of the values is dependent on desired location of the pick-off point on the block diagram. The final result is not altered.